Display
Stacks

lume IX October 1955 Number 52

UNIVERSITY
OF MICHIGAN
FEB 16 1956
MATH. ECON.
LIBRARY

43

# Mathematical Tables

## *and other*

# Aids to Computation

**Published Quarterly**

*by the*

**National Academy of Sciences—National Research Council**

Published quarterly in January, April, July, and October by the National Academy of Sciences—National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

---

**Editorial Committee**
**Division of Mathematics**
**National Academy of Sciences—National Research Council**
**Washington, D. C.**

C. B. TOMPKINS, *Chairman*, University of California, Los Angeles, California

J. H. BIGELOW, The Institute for Advanced Study, Princeton, New Jersey

C. C. CRAIG, University of Michigan, Ann Arbor, Michigan

A. ERDÉLYI, California Institute of Technology, Pasadena, California

A. H. TAUB, University of Illinois, Urbana, Illinois

J. TODD, National Bureau of Standards, Washington, D. C.

• • •

## Information to Subscribers

The journal is now published quarterly in one volume per year with issues numbered serially since Volume I, Number 1. Subscriptions are $5.00 per year, single copies $1.50. Back issues are available as follows:

Volume I (Nos. 1–12, 1943–1945) $12.00 (sold only as a complete volume)

Volume I (No. 7, 1944) "A Guide to Tables of Bessel Functions," by H. Bateman and R. C. Archibald, 104 pp., $2.00

Volume II and III (Nos. 13–28, 1946–1949) $4.00 per year, $1.25 per issue

Volume IV and following, $5.00 per year, $1.50 per issue

All payments are to be made to the National Academy of Sciences and forwarded to the Publications Office, 2101 Constitution Avenue, Washington, D. C.

Agents for Great Britain and Ireland: Scientific Computing Service, Ltd., 23 Bedford Square, London W.C.1

• • •

## Information to Contributors

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation* and all books for review should be addressed to C. B. Tompkins, Department of Mathematics, University of California, Los Angeles 24, California. The author should mention the name of an appropriate editor for his paper if this is convenient.

# Polynomial Root Solving on the Electronic Differential Analyser (A Technique for Finding the Real and Complex Roots of a Polynomial using an Electronic Differential Analyser)

**1. Introduction.** This paper presents a technique for the solution of high degree polynomials with real coefficients. By means of this technique the real and complex roots of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_nx^2$$

can be obtained. The technique is essentially a graphic one in which the function $f(x)$ and its derivatives $f'(x)$, $f''(x)$, $\cdots$ are generated by an electronic differential analyser [1]. The method depends heavily on the ability of the analyser to produce an accurate plot of the function and its derivatives. The use of these accurate graphs is obvious for finding the real roots of the equation. But a systematic use of graphs for the finding of complex roots is not so obvious and hence not widely used [2].

This computer approach to the finding of the complex roots requires additional curves of the real and complex parts of the complex function

$$U + iV = f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n; \quad (z = x + iy).$$

The real part $U(x, y)$ and the complex part $V(x, y)$ are also plotted by the analyser and complex roots are found from the common zeros.

**2. Method of Solution.**

*1. Real Roots.* In order to generate the equation $y = f(x)$ on the differential analyser the polynomial must be differentiated $n$ times, where $n$ is the degree of the polynomial. This will produce the derivatives $f^n(x)$, $f^{n-1}(x)$, $\cdots$, $f'(x)$ where $f^n(x)$ is a constant, say $f^n$. If a voltage of value $f^n$ is introduced into an integrating amplifier the output is $xf^n$. (See Fig. 1a.) Putting the value of $f^{n-1}(0)$ into the same integrator makes the output $f^{n-1}(x)$ (neglecting the sign change of the amplifier). This process is continued until the final integration produces $f(x)$. The real roots are picked from the curve as the values of $x$ for which $f(x) = 0$.

*2. Complex Roots.* The technique for finding the complex roots of the polynomial is based on the idea of a surface $f(z)$ over the horizontal $xy$ plane where

$$f(z) = f(x + iy) = U(x, y) + iV(x, y).$$

For $f(z)$ real, $V(x, y) = 0$; $f(z) = U(x, y)$ so $U(x, y)$ is a space curve whose projection on the $xy$ plane is $V(x, y) = 0$.

For the solution of the polynomial equation $f(z) = U + iV = 0$ both the real function $U$ and the imaginary function $V$ must be zero so the root $x_1 + iy_1$ represents the intersection of the space curve $U(x, y)$ with the curve $V(x, y) = 0$. This is a point in the $xy$ plane where the $U$ function is zero.

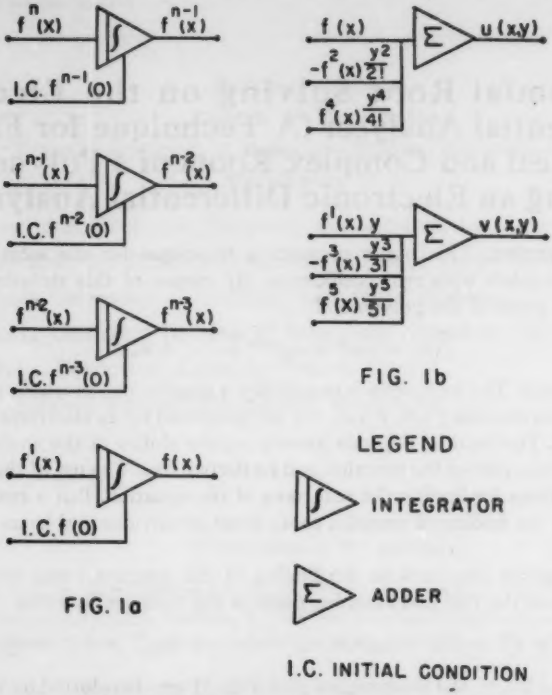



FIG. 1. General wiring diagram for polynomial root solving. Sign inverters have not been shown.

The differential analyser can handle only one variable of integration so the common point in the $xy$ plane where both $U$ and $V$ are zero must be approached by a judicious selection of one of the variables, usually $y = y_1$. Then a plot of the two functions $U(x, y_1)$ and $V(x, y_1)$ reveals how close $y_1$ is to the actual imaginary part of the root. If the plot of $U_1$ and $V_1$ indicates a common crossing of the axis, say at $x_1$, then the chosen $y_1$ is the correct one and the root is $x_1 + iy_1$. Notice that this also yields the conjugate root $x_1 - iy_1$.

If the curves $U_1$ and $V_1$ do not have a common crossing of the abscissa, a change in $y_1$ is made and another plot of the two equations is generated. This is continued until the root is obtained to the desired accuracy.

The judicious choice of $y_1$ is assisted by a knowledge of the positions of maxima, minima, points of inflection, and accessory points of inflection (where a maximum and a minimum have run together) of the real curve.

A. J. KEMPNER'S [2] suggestion that a study of the derivatives is helpful in estimating the complex roots of a polynomial is borne out by the nature of the real and imaginary parts of the complex function $f(z)$. The real part

$$U(x, y) = f(x) - f''(x)y^2/2! + f^{(IV)}(x)y^4/4! - \cdots,$$

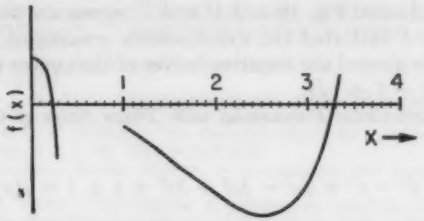



FIG. 2. Graph showing real roots of quintic equation $x^5 + x^4 - 7x^3 - 22x^2 + x + 1 = 0$ of example 1. Note: Change of vertical scale at $X = 1$.

and the imaginary part

$$V(x, y) = f'(x)y - f'''(x)y^3/3! + f^{(V)}(x)y^5/5! - \cdots;$$

that is, the real part is the alternating sum of the polynomial and its even derivatives multiplied by the corresponding even powers of $y$ and inverse factorials, while the imaginary part is the sum of the odd derivatives multiplied by the odd powers of $y$ and corresponding inverse factorials.

Since the respective derivatives are generated in order to produce the function $f(x)$ (as shown in Fig. 1a) they are available for finding a first approximation to the imaginary part of a complex root. They are then used for producing $U(x, y_1)$ and $V(x, y_1)$ after being multiplied by the appropriate factor (e.g., $f''(x)$ is multiplied by $y_1^2/2!$). (See Fig. 1b.)

**3. Examples of Computer Solutions.**

*Example 1. Real Roots of a Quintic Equation.* The polynomial

$$x^5 + x^4 - 7x^3 - 22x^2 + x + 1 = f(x)$$

is used as an example for the machine solution of real roots. The roots sought are the real positive roots [3]. This requires only the generation of the respective derivatives (Fig. 1a) to produce $f(x)$. The computer plotted the function as shown in Fig. 2. The curve crossed the $x$ axis at the roots 0.22 and 3.28.

*Example 2. Complex Roots of a Quartic Equation.* The polynomial

$$x^4 + x^3 + 5x^2 + 5x + 12 = f(x)$$

is used as an example for the machine solution of complex roots [4]. The computer

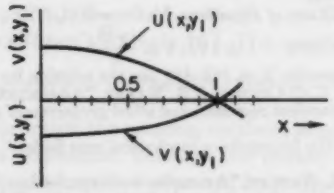



FIG. 3. Graph showing common intersection of $x$ axis by $U(x, y_1)$ and $V(x, y_1)$ indicating the roots $-1 \pm i\sqrt{2}$ of the equation $x^4 + x^3 + 5x^2 + 5x + 12 = 0$.

is wired as in Fig. 1a and Fig. 1b and $U$ and $V$ curves are plotted for negative $x$ with $y = \sqrt{2}$. Fig. 3 indicated the simultaneous crossing of the axis at $x = 1$. Since the computer plotted the negative halves of the curves this crossing is then the conjugate pair $-1 \pm i\sqrt{2}$.

*Example 3. Sixth Order Polynomial with Three Pairs of Complex Roots.* The polynomial [5]

$$x^6 - x^5 + 2x^4 - 3x^3 + 2x^2 + x + 1 = f(x)$$

was chosen as a fairly difficult equation to solve by this method since all roots are complex and clustered about the origin.

Figure 4 is a plot of the expressions $U$ and $V$ for the value of $y_1 = 0.64$. The pair of roots as given by the computer was $1.0 \pm 0.64i$. (These roots are given in USPENSKY as $1.08018 \pm 0.63903i$.)

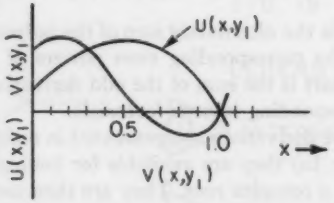



FIG. 4. Graph showing common intersection of $x$ axis by $U(x, y_1)$ and $V(x, y_1)$ indicating the roots $1.08 \pm 0.64i$ of the equation $x^6 - x^5 + 2x^4 - 3x^3 + 2x^2 + x + 1 = 0$.

**4. Conclusion.** The method of root solving presented herein is a new technique employing the latest type of analog computer to the old problem of root solving. The solutions generated by the computer are fast as compared with the computational methods or the mechanical root finders. The accuracy of roots can easily be kept in the range of engineering accuracy. Mathematical manipulations of the original equation are minimized by the method which requires only the derivatives of the polynomial whose roots must be determined.

CYRIL ATKINSON

University of Calif.
Berkeley, Calif.

1. The electronic differential analyser used is the Beckman EASE computer located at the Timoshenko Laboratory, Stanford University.

2. A. J. KEMPNER, "On the Complex Roots of Algebraic Equations," Amer. Math. Soc., *Bull.*, v. 41, 1935, p. 809–843.

3. J. V. USPENSKY, *Theory of Equations*, McGraw-Hill, N. Y., 1948.

4. See reference 3 for roots $-1 \pm i\sqrt{2}$; $\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$.

5. See reference 3 appendix V, p. 318–331, for the solution by GRAEFFE's method.

6. A. R. BOOTHROYD, E. C. CHERRY, & R. MAKAR, "An electrolytic tank for the measurement of steady-state response, transient response, and allied properties of networks," I.E.E., *Proc.*, v. 96, Part I, 1949, p. 163–177.

7. R. L. DIETZOLD, "The Isograph—a mechanical root finder," *Bell Laboratories Record*, v. 16, 1937–38, p. 130.

8. G. FERRARA & R. L. NADEAU, "A complex wave synthesizer," *Elec. Eng.*, v. 70, 1951, p. 585.

9. J. S. FRAME, "Machines for solving algebraic equations," *MTAC*, v. 1, 1945, p. 337–353.

10. H. C. HART & I. TRAVIS, "Mechanical solution of algebraic equations," *J. Franklin Inst.*, v. 215, 1938, p. 63–72.

11. A. J. KEMPNER, "Geometry as an avocation," *Amer. Math. Mon.*, v. 40, 1933, p. 455–471.
12. BYRON O. MARSHALL, JR., "The electronic isograph for roots of polynomials," *J. Appl. Phys.*, v. 21, 1950, p. 307–312.
13. J. H. ROBERTSON, "A simple machine capable of Fourier synthesis calculation," *J. Sci. Instr.*, v. 27, 1950, p. 276–278.
14. A. RUSSELL & J. N. ALTY, "An electromagnetic method of studying the theory of and solving algebraic equations of any degree," *Phil. Mag.*, s. 6, v. 18, 1909, p. 802–812.
15. T. B. RYMER & C. C. BUTLER, "An electrical circuit for harmonic analysis and other calculations," *Phil. Mag.*, s. 7, v. 35, 1944, p. 606–616.
16. WALTER W. SOROKA, *Analog Methods in Computation and Simulation*, McGraw-Hill, New York, 1954.
17. L. BAUER & S. FIFER, "The solution of polynomial equations on the REAC," *Project Cyclone Symposium One*, Reeves Instrument Corp., New York, 1951, p. 31–36.
18. F. W. J. OLVER, "The evaluation of zeros of high-degree polynomials," Royal Soc. of London, *Phil. Trans.*, ser. A, v. 244, 1952, p. 385–415.
19. R. A. BROOKER, "The solution of algebraic equations on the EDSAC," Cambridge Phil. Soc., *Proc.*, v. 48, 1952, p. 255–270.

# Notations for Partitions

The present paper gives a contribution to the automatic calculation of the characters of certain symmetric groups, as described in a paper which recently appeared in *MTAC* [1]. The repeated application of the recursion formula (7) of the cited paper involves the storing of a chain of partitions. The allotted storage space must be sufficient to meet even the case of a chain of as many as about $n/2$ partitions. In other words, provision must be made for storing about $n^2/4$ integers, components of these partitions. It would be a waste of memory space to store these integers separated, numerous and small as they are. Furthermore, as $n$ increases, it will be still more important that the partitions should be available in the rapid access storage of the machine. Therefore, they should be stored by some method of packing, suitable for the operations to be applied, particularly during the calculation of characters.

In the sequel, I shall briefly describe three such methods, which might perhaps be of some interest. The first method [2] was used in an experimental program for character calculation with the Swedish relay computer BARK (August, 1953) and also in the first version of the character program using the Swedish electronic computer BESK (Spring, 1954). The second method is employed in the actual version of this program (tested on BESK, June, 1954). The third method, a modification of the second, has not yet been used in the program, though it is described here for theoretical reasons.

**Preliminaries.** Theoretically, the minimum of the number, $x$, of binary digits required to represent all the partitions of an integer $n$, is the smallest integer, not less than $\log_2 P(n)$, where $P(n)$, the number of the partitions, can be estimated by means of HARDY-RAMANUJAN's formula [3]. The result obtained,

$$x_{\min} \sim \pi\sqrt{2/3}\,\log_2 e\cdot\sqrt{n} - \log_2 4n\sqrt{3} \sim 3.70\sqrt{n} - \log_2 n - 2.79$$

indicates an ideal towards which the packing methods should lead.

If packing with fixed boundaries between the packed elements, one should store the partition components, each diminished by 1. In this case, $n$ must also be stored. The value of $x$ then lies between $n$ and $3n/2$. For example, in a 40 digit register one can store the partitions of $n \leq 30$ by this method. It will be seen that

our first and second methods yield slightly better values of $x$, namely $x = n$ and $x = n + 1$, respectively, whereas our third method gives approximately $x \leq \sqrt{32n + 4} - 2$, which is, for $n > 28$, in better accord with the Hardy-Ramanujan value.

Our notations for partitions are closely related to the YOUNG diagrams of the partitions [4], which are important in the NAKAYAMA version [5] of the recursive method. This version simplifies the calculation scheme, and it has, therefore, been employed in our program.

It is essential, in the methods which we are going to describe, that the computing machine is equipped with the operation known as "normalization [6]." We assume that the machine represents numbers modulo 2 by $N$ digit words, the fractional part consisting of $N - 1$ binary digits, the remaining binary digit being interpreted as sign digit. Because it is necessary to store the sign associated to each partition appearing in the recursion formula, we shall represent the partition itself entirely in the fractional part of a word.

**First method.** The partition $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ is represented by the number

$$L = 2^{-\lambda_1} + 2^{-\lambda_1-\lambda_2} + \cdots + 2^{-\lambda_1-\lambda_2-\cdots-\lambda_p};$$

i.e., starting from the binary point, $(\lambda_1 - 1)$ zeros followed by a 1, $(\lambda_2 - 1)$ zeros followed by a 1, *etc.* The last 1, which means $2^{-n}$, is followed by zeros (if $n < N - 1$). The number $n$ need not be stored separately. For example, the partition (5, 2, 2, 1, 1, 1) is represented, the sign digit omitted, by the fraction

$$L = .000010101111 \text{ (and then zeros).}$$

We use shifts to read in the components of $(\lambda)$, normalizations to read them out [2].

If the partitions of $n$ are enumerated in decreasing lexicographical order, their representing numbers $L$ form an increasing series. Moreover, the transition from one partition to the next is effected by the addition of $2^{-n+1}$ to $L$, followed by a slight modification if necessary.

This method is still used in our character program in the routine for passing from one partition to the next one, when several characters are to be calculated in succession.

**Second method.** This time we write the partition

$$(\lambda) = \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_m^{a_m}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and all $a_i > 0$, and put $\lambda_i - \lambda_{i+1} = b_i$ $(i < m)$, $\lambda_m = b_m$. The positive integers $a_i$ and $b_i$ $(i = 1, 2, \cdots, m)$ define the partition completely. The partition is represented by a number $L'$ constructed in the following manner. Denoting by $c$ an arbitrary, non-negative integer (usually, $c = 0$), yet with the restriction $c + a_1 + b_1 + \cdots + a_m + b_m \leq N - 1$, there are in $L'$, after the binary point, $c$ ones (yet usually, $c = 0$), and then successively $a_1$ zeros, $b_1$ ones, $a_2$ zeros, $b_2$ ones, *etc.* Eventually, remaining space after the last $b_m$ ones is filled by zeros.

Again, we read in and out with the aid of shifts and normalizations.

Each non-vanishing term in the recursion formula is due to the existence of a so-called "hook [5] of length $h$" in the Young diagram, $h$ being the actuated component of the class partition. Such hooks are most easily recognized in the number $L'$, namely as a pair of a 0 and a 1, spaced $h$ binary places from each other, the 0 being the most significant digit in that pair. The sign of the corresponding term of the formula is $(-1)^s$, where $s$ is the number of zeros between the 0 and the 1 which were said to form the pair. The removal of the hook is an operation to be performed before the recursion formula is used again. It is equivalent to the interchange of the two digits of the mentioned pair.

These operations, namely the searching for hooks, the determination of signs, and the removal of hooks, form the essential contents of the calculation scheme. We note that the removal of a hook will never cause an increase of the number of digits of $L'$. Finally, the conjugate partition $(\bar{\lambda})$ is very easily found, being represented by the "complement on 1" of the number obtained by reading $L'$ in the reverse order. BESK has an instruction for shifting the multiplier register into the accumulator in reversed order.

**Third method.** The methods just described imply that an integer, $t$, will be represented by $t$ digits. The fact that each digit is capable of two values is used only to realize the boundaries between adjacent integers. A more efficient notation is obtained as follows. The integer $t$, which is to be packed together with other integers, is first written in binary notation. The number, $\tau$, of requisite digits is then represented by $\tau$ digits, all ones or all zeros. The two parts of the representation thus obtained are most conveniently placed in the different halves of one and the same whole word, $\tau$ being placed in the half word containing the binary point. Thus the normalization, being guided by $\tau$, gives us the value of $t$ in binary notation, unpacked.

It should be observed that such symbols as 1, 01, 001, etc., can be used here to represent different values of $t$. In this way, the integers $t = 1, 2, 3, 4, 5, 6, 7$, etc., will be represented by the symbols 0, 1, 00, 01, 10, 11, 000, etc., respectively, the corresponding values of $\tau$ being 1, 1, 2, 2, 2, 2, 3, etc. Therefore, if $\tau$ is determined so that

$$2^\tau - 1 \leq t < 2^{\tau+1} - 1,$$

the number $t$ will be written in $\tau$ binary digits, constituting a number usually read as $t + 1 - 2^\tau$.

This "half-binary" notation could be used for packing the integers $c, a_1, b_1, \cdots, a_m, b_m$, appearing in the second method. If the numbers of digits in the binary notations of these integers are $\gamma, \alpha_1, \beta_1, \cdots, \alpha_m, \beta_m$, respectively, we write $\gamma$ ones, $\alpha_1$ zeros, $\beta_1$ ones, *etc.*, in the binary point half word, and the binary notations of $c, a_1, b_1$, *etc.*, in the remaining half word. This gives us a notation for partitions requiring a number of digits roughly proportional to $\sqrt{n}$, as formerly indicated.

Of course, the method could be iterated (writing $\tau$ binarily, *etc.*), but this would not give a very efficient notation for partitions.

**Other applications.** As an example, we indicate how the methods can be applied to the storing of the factorization of an integer, $n$, into powers of primes. Let $p_i$ $(i = 1, 2, \cdots, m)$ be the primes dividing $n$, $p_1 > p_2 > \cdots > p_m \geq 2$, and

let $a_i$ be the exponent denoting the highest power of $p_i$ still dividing $n$, $a_i > 0$. Further, let the function $f(x)$ mean the number of all existing primes $\leq x$. Putting $f(p_i) - f(p_{i+1}) = b_i$ $(i < m)$, $f(p_m) = b_m$, we can use the second or third method above to store the numbers $a_1, b_1, \cdots, a_m, b_m$. These determine the factorization completely.

STIG COMÉT

Matematikmaskinnämnden
(The Swedish Board for Computing Machinery)
Box 6131, Stockholm 6, Sweden

1. R. L. BIVINS, N. METROPOLIS, P. R. STEIN, & M. B. WELLS, "Characters of the symmetric groups of degree 15 and 16," *MTAC*, v. 8, 1954, p. 212.
2. S. COMÉT, "On the machine calculation of characters of the symmetric group," Tolfte Skand. Matematikerkongressen, Lund, 1953, *Comptes rendus*, p. 18–23.
3. G. H. HARDY & S. RAMANUJAN, "Asymptotic formulae in combinatory analysis," London Math. Soc., *Proc.*, s. 2, v. 17, 1918, p. 75.
4. A series of eight papers by A. YOUNG, "On quantitative substitutional analysis," published in London Math. Soc., *Proc.*, beginning in ser. 1, v. 32, 1901, p. 384, and ending in ser. 2, v. 37, 1934, p. 441.
5. T. NAKAYAMA, "On some modular properties of irreducible representations of a symmetric group," *Jap. J. Math.*, v. 17, 1941, p. 165 and p. 411.
6. This involves shifting to the left until the most significant digit is 1, and counting the shifts; see G. KJELLBERG & G. NEOVIUS, "The BARK, a Swedish general purpose relay computer," *MTAC*, v. 5, 1951, p. 29–34.

# Determination of Steiner Triple Systems of Order 15

**1. Introduction.** A Steiner Triple System is a set of subsets of $n$ marks such that: a) Each subset contains three distinct marks, b) each pair of distinct marks appears in one and only one subset. It is easy to see that $n \equiv 1$ or 3 (mod 6) [1] and that the number of subsets is $n(n-1)/6$. For $n = 3, 7, 9$ there is one and essentially only one such set, all others being obtained by permutation of the marks. For $n = 13$, there are two distinct systems.

Using SWAC for the computations we have obtained the non-isomorphic systems of order 15. During the course of the calculations, Prof. L. J. PAIGE called our attention to a series of papers [2] which culminate in the listing by F. N. COLE of a set of 80 systems of order 15 with proof that these are distinct and an argument tending to show that the listing is exhaustive. Since following the argument would entail tremendous hand calculation and since the number 80 was below our expectation so that it seemed quite possible that there had been errors or omissions in the determination, we decided to continue. It turns out that these 80 are indeed complete and that Cole's computations were either correct or at any rate did not contain any errors which led to the omission of a valid system. In any case these results testify to Cole's reputation for brilliant computation.

**2. The computation—General considerations.** There are two stages in the calculation, the formation of the systems and the reduction of isomorphisms. That these must be carried out simultaneously is seen from the fact that if all systems on 15 marks which arise from the 80 distinct systems are counted, the total exceeds $6.10^{13}$ (actually (14521/315)15!).

In practice the reduction was carried out at three places. First the total number of systems to be computed by SWAC was reduced to the order of $10^4$ by giving the machine partial systems to complete in such a way that all distinct systems were certain to be included in the final result. Second, in the later stages of the computation it was possible to inhibit output of computed systems by a simple test for isomorphism to previously determined systems which could be added to the completion routine. Finally the output systems had to be tested for permutations taking them into other outputs.

**3. Initial data.** It is clear that, given any Steiner Triple System on fifteen marks, by renaming we may assume that the triples $(1, 2, 3)(1, 4, 5)(1, 6, 7)$ $(1, 8, 9)(1, 10, 11)(1, 12, 13)(1, 14, 15)(2, 4, 6)$ are present. It is then easy to see that there are essentially only four ways to complete the triples containing 2. These are:

A: $(2, 5, 7)(2, 8, 10)(2, 9, 11)(2, 12, 14)(2, 13, 15)$
B: $(2, 5, 7)(2, 8, 15)(2, 9, 10)(2, 11, 12)(2, 13, 14)$
C: $(2, 5, 8)(2, 7, 9)(2, 10, 12)(2, 11, 14)(2, 13, 15)$
D: $(2, 5, 8)(2, 7, 10)(2, 9, 12)(2, 11, 14)(2, 13, 15)$.

Cole calls these respective arrangements, triple tetrad, single tetrad, hexad, and duodecad, since the first has three sets of four triples in the form: $(1, \alpha, \beta)$ $(1, \gamma, \delta)(2, \alpha, \gamma)(2, \beta, \delta)$. The second has only one such tetrad and the third has two sets of the form $(1, \alpha, \beta)(1, \gamma, \delta)(1, \epsilon, \zeta)(2, \alpha, \gamma)(2, \beta, \epsilon)(2, \delta, \zeta)$. D has no combination of interlocking triples short of the full subsystem with the obvious omission of $(1, 2, 3)$.

Again, it is apparent that, in case A, we may assume the presence of either $(3, 4, 7)$ or $(3, 4, 8)$. For suppose the triple $(3, 4, k)$, $k > 8$; there is a set of four triples $(1, k, j)(1, m, n)(2, k, m)(2, n, j)$; the permutation $(8, k)(9, j)(10, m)(11, n)$ will produce the triple (348) leaving the remainder invariant. Further, in the case (347) we may assume the remaining triples including 3 to be $(3, 5, 6)$, $(3, 8, 11)$ $(3, 9, 10)(3, 12, 15)(3, 13, 14)$ for if, say, $(3, 5, 8)$ occurred, the permutation $(45)(67)$ will leave the triples on 1 and 2 invariant while introducing (348).

In a similar fashion the cases considered under B, C, and D, may be reduced. In this reduction the automorphism group on the partial system plays an important role. Finally, we may consider the sets in a hierarchy. Under B, no system which includes triple tetrads on the triples using any pair of marks $(a, b)$ to replace $(1, 2)$ in the definition of triple tetrad above need be considered since a permutation of the form $(1, a)(2, b)\cdots$ would reduce it to one of the A systems. Under C no system with a tetrad comes into consideration. Under D we can rule out any system containing either a hexad or a tetrad. This consideration actually serves to exclude D completely.

The length to which this procedure is carried must be balanced between several conflicting requirements:

1) The machine should produce as few systems as possible.

2) There should be as few starts as possible since each new subsystem demands new feeding of initial data which is time-consuming to prepare.

3) The initial hand computation must not be carried to a point where there is danger of losing completeness by errors arising from its complexity.

The requirement 3) is less important than it appears at first glance since it is far easier to neglect existing automorphisms (which would create over-production) than to introduce non-existing ones. At any rate 2) and 3) together determined that no production of initial data beyond triples on 3 was considered. The initial starts, then, included from 14 to 19 fixed triples. For example, in the case A: (348) the following additional triples were used: 1. (3, 5, 6), 2. (3, 5, 9)(3, 12, 15)(3, 13, 14), 3. (3, 5, 10)(3, 12, 15)(3, 13, 14), 4. (3, 6, 12). On start 1, the machine was stopped after (3, 7, 9), (3, 7, 10), (3, 7, 11), had been considered.

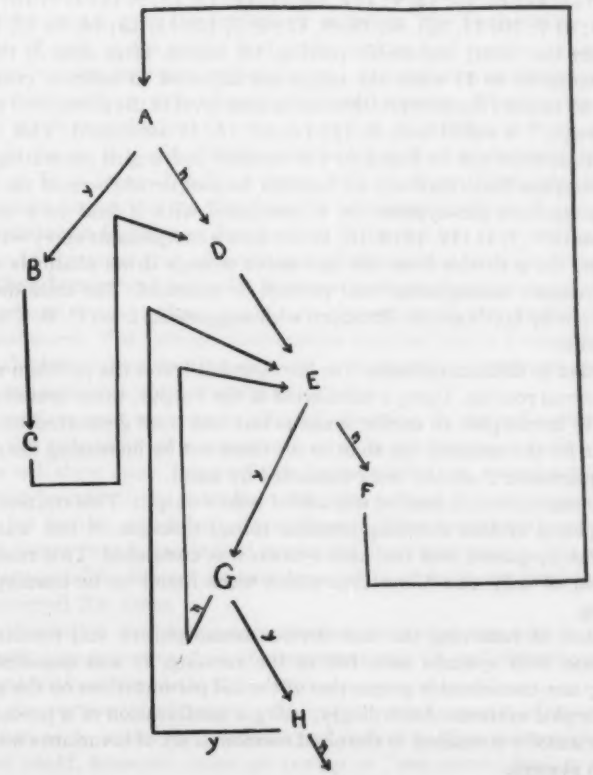
**4. Exhaustive production of Steiner Triple Systems.** As remarked the computations were performed on SWAC. This is a binary machine using 36 digits plus sign in a word. The high speed memory consists of 256 words and the medium speed memory was a drum holding 4096 words. These sizes limited the choice of routines and the methods of storage.

The machine is required to form all systems using the given first triples. Rather than a list of triples the machine actually stores and works with an incidence or usage table. Specifically, if we designate the first four 8 digit portions of a SWAC word by $\alpha$, $\beta$, $\gamma$, $\delta$, each mark is stored in $\alpha$ of a corresponding word; $\beta$ holds the number of marks with which the given mark has appeared at that stage of the computation; the $i$th position of the first fifteen digits of $\gamma$ and $\delta$ holds a 1 if the given mark has been used with $i$ and a 0 otherwise. Thus the cell corresponding to the mark 8 in the entry A: (348) case, initially read in decimal: $\alpha = 008$, $\beta = 007$, $\gamma = 241$, $\delta = 192$. Note that an incidence of a mark with itself is always counted so as to instruct the machine unambiguously that a mark $i$ is available for use in a triple with a mark $j$ if and only if a 0 appears in the $i$th place of $(\gamma, \delta)$ in the word corresponding to $j$. With this and the binary representations of decimal 241 (1111, 0001) and decimal 192 (1100, 0000), it is immediately seen that the triples (1, 8, 9)(2, 8, 10)(3, 4, 8) generate the line.

The basic code may be outlined as follows:

| *Routine* | *Possible Predecessors* | *Operations Performed* | *Successor Criterion* | *Yes* | *No* |
|---|---|---|---|---|---|
| A | F | Find least $\alpha$ whose $\beta < 15$ | Are all $\beta \geq 15$? | B | D |
| B | A | Output (or go to supplementary routine for search for earlier isomorphs) | No choice except possibility of inserting supplementary routine | C | C |
| C | B or supplementary routine | Return to next earlier value of $\alpha$ | No choice | D | D |
| D | A or C | Determine marks available for use with present choice of $\alpha$ value | No choice | E | E |

| *Routine* | *Possible Predecessors* | *Operations Performed* | *Successor Criterion* | *Yes* | *No* |
|---|---|---|---|---|---|
| E | D | Permute to form new subsystem on mark denoted by present value of $\alpha$ | Is a new subsystem found? | F | G |
| F | E | Store new subsystem with other completed ones | No choice | A | A |
| G | E | Have all permutations for this $\alpha$ been tried? | Have all permutations for this $\alpha$ been tried? | H | E |
| H | G | Recall next earlier $\alpha$ and its list | Is an earlier $\alpha$ available? | E | I |
| I | H | Halt (end of calculation) | | | |



The permutation scheme of routine E is a modification of a Paige-Tompkins routine [3] adapted to the fact that order within a triple is of no importance and thus there are only $(n-1)(n-3)\cdots 1$ permutations on $n$ marks to be tried. Both the permutation scheme and the complete code can be illustrated by the example A: (348)(356). Here 1 and 2 are complete, their $\beta$ is 15. For 3, 4, 5, 6, 8, $\beta = 7$. For 7, 9, 10, 11, 12, 13, 14, 15, $\beta = 5$. Thus 3 is selected as the least number still incomplete with largest possible $\beta$. The numbers 7, 9; 10, 11; 12, 13; 14, 15; are available for use with 3, the semicolons indicating the initial breakdown into pairs which could form triples with 3. Permutation occurs as far to the left as possible, either because of impossible pairings or fully utilized sub-systems. Here (10, 11) is the first impossible one. The next considered is 7, 9; 10, 12; 13, 14; 15, 11. This is acceptable and the lines of the table are stored in this order on a drum channel; the permutation tallies are also stored for future reference. The incidence tables are then adjusted in the high speed memory to account for this sub-system and we proceed. This time 4 is selected with its list of *7, 9*; 10, 11; 12, 13; 14, 15. The successive trials from this point to form a new sub-system are: 1) 7, 10; 11, 12; *13, 14*; 15, 9; 2) 7, 10; 11, 12; *13, 15*; 9, 14; 3) 7, 10; 11, 12; 13, 9; *14, 15*; 4) 7, 10; 11, 12; *14, 15*; 9, 12; 5) 7, 10; 11, 13; 14, 9; 12, 15. The italics indicate the (first) impossible pairing. Of course, after step 3) the order returns momentarily to 1) while the tallies are adjusted to indicate completion of this level and to start the permutation at the next level in the prescribed pattern.

After storage, 7 is called with 8, 11; *12, 13*; 14, 15 associated. This time no acceptable sub-system can be found so 4 is recalled and begins permuting where it left off. It happens that this entry on 3 cannot be completed; none of the twenty apparently acceptable sub-systems on 4 associated with it lead to a complete system. Not until 3; 7, 11; 12, 15; 9, 10; 13, 14 does a completable entry on 3 arise.

To output, the $\alpha$ entries from the successive storage drum channels are collected in a compact arrangement and printed or punched. The code described here was written by LaVerne E. Rickard with suggestions from C. B. Tompkins and the authors.

**5. Reduction to distinct systems.** For the A and B levels the problem was run with no additional routine. Using a tabulation of the output, those systems which were obviously isomorphic to earlier systems but had been generated because it was simpler to let the machine run than to cut them out by increasing the number of starts (requirement 2 above) were discarded by hand.

In the C computation, a routine was added before output. This routine looked at each completed system checking possible tetrad linkages. If one was found the output was by-passed and the next system was computed. This resulted in the production of only two C-systems which were found to be isomorphic on hand-checking.

The problem of removing the non-obvious isomorphisms still remained. At this point some 5000 systems were left in the running. It was manifestly impossible to try any considerable proportion of the 15! permutations on the possible pairs of isomorphic systems. Accordingly, using a modification of a procedure of Louise D. Cummings contained in the cited memoir, a set of invariants was tabulated for each system.

Given a triple system, the sub-system consisting of the triples on any mark possesses one of the linkage types A, B, C, D, with the sub-system on each of

the 14 other marks. To each mark, then, can be correlated a vector $(a, b, c, d)$ where $a + b + c + d = 14$; $a$ is the number of A linkages with the given mark, $b$ the number of B linkages, etc. Thus associated with each system is a $15 \times 4$ matrix and if two systems are isomorphic their matrices are row permutations of each other. Further, if system $S_1$ is carried into system $S_2$ by the permutation $\begin{pmatrix} 1 & 2 & \cdots & 15 \\ i_1 & i_2 & & i_{15} \end{pmatrix}$, the vector corresponding to $j$ of $S_1$ must be equal to the vector corresponding to $i_j$ of $S_2$. A routine was written for SWAC which computed these vectors (and their sum for sorting purposes) for each system. When a system was input, a table was first formed which stored in the $(i, j)$ position the number $k$ appearing in the triple $(i, j, k)$ of the system. Zero was stored in the $(i, i)$ positions. Since $4 \cdot 15 = 60$ digits were required for any line of the table, double precision computation was necessary. The main part of the routine then took each ordered pair $(i, j)$ and found the coincidences. If there were three tetrads a tally was added in the $\alpha$ section of the $i$th output word, etc. In working with the B systems a rejection step was added whenever a triple tetrad occurred.

There now remained 1912 A-systems and 538 B-systems which were sorted into stacks according to the equivalent matrices. The final routine used a representative of each of the stacks and attempted to find a permutation taking the successive systems into this archetype; i.e., choices were made for $i_1$ and $i_2$ from the possible numbers with correct associated vectors. This determined $i_3$ from (1, 2, 3) which was checked for vector equality. If it did not check, a new choice was made for $i_2$; if it did check, a choice for $i_4$ was made which determined $i_5$ from (1, 4, 5), $i_6$ from (2, 4, 6), $i_k$ from $(34k)$, $i_m$ from $(35m)$, etc. Each of these was checked for vector equality and for consistency. (E.g., if $m = 6$, $i_6 = i_m$.) The permutation found was then output. It turned out that the invariants were complete in the sense that all systems with equivalent matrices were isomorphic.

**6. Time factors and controls.** Systems were produced at rates varying from 20 per minute down to three per minute depending on the relative "richness" of the initial starts. The linkage computation required about 2 seconds per system. The final permutation routine required an average of 4 seconds per system including typewriter output of the permutation.

All routines used were essentially self-checking. In the exhaustive production routine the possibility of a consistent error in all the usage tables is negligible. An error will show up as "$i$ is available for use with $j$ but $j$ is not available with $i$." When this happened the machine went into a short loop and could only be restarted by feeding consistent data. Several hand computational checks were made to insure that it was indeed producing the "next system." It may be noted that every possible distinct system occurred in at least two isomorphs. Some of them occurred 200 times.

The checks on the vector computation were that the matrix obtained was symmetric, that every vector had component sum 14, and that the matrices were consistent with others of the same type.

For the permutation routine, the various requirements of consistency were again such as to make the possibility of finding a false permutation negligible. An error could, however, cause an answer of "not isomorphic" to appear. This did, in fact, happen in several cases. In each case the system was re-fed twice; the resulting permutations were compared and hand checked.

**7. Conclusions.** The chief source of wasted time lay in the lack of initial knowledge concerning the magnitude of the problem:

a) The original output of the Rickard code was in converted decimal, designed for easy tabulation and hand procedures. These outputs had to be reconverted into binary form readily assimilated by SWAC.

b) Systems with A-linkage should have been rejected without output during the B-computation as was done for tetrad linkage in the C-computation. In fact, the linkage vectors should have been computed simultaneously with the system. However, we had not thought of these matrices until these calculations were complete.

The procedures described in this paper represent, to the best of our knowledge, the first use of large scale digital computing equipment to isolate isomorphic algebraic systems. In this connection, the crucial step is the devising of a reasonably calculable system of invariants to reduce the gross search for mappings. By its nature, a computer is better adapted to calculating such invariants than to exhaustive search for transformations.

The question of computing the systems of order 19 arises. Here there are seven possible starts on the systems using 2; these may be represented by the possible partitions of 16 into even numbers greater than 2 in the same way that A, B, C, D, in the fifteen case correspond to $4+4+4$, $4+8$, $6+6$, 12 respectively. For each of these starts there are $2 \cdot 10^6$ *a priori* distinct permutations of $(4, 5, \cdots, 19)$ for use with 3 compared to $10^4$ on $(4, 5, \cdots, 15)$. Machine procedures would be required to isolate the starts and the total number requiring trial might be as great as $10^4$. The problem thus seems to be beyond the reach of present equipment. Some short runs of systems have been made on SWAC and these bear out this impression. Unfortunately, no reliable estimate of the total number of systems to be expected exists.

The authors wish to acknowledge the assistance given them in coding and computation by C. B. Tompkins, L. J. Paige, Ruth B. Horgan, LaVerne E. Rickard, and John Selfridge. The codes used are on file at Numerical Analysis Research, University of California, Los Angeles.

MARSHALL HALL, JR.
J. D. SWIFT

Ohio State University
University of California, Los Angeles,
and Institute for Advanced Study

The computations on SWAC were sponsored by the Office of Naval Research and the Office of Ordnance Research. The preparation of this paper has been sponsored in part by ONR Contract NR 045 141.

1. It has been known since 1859 that this necessary condition is also sufficient for the existence of a Steiner triple system. M. REISS, "Ueber eine Steinersche combinatorische Aufgabe," *J. Reine Angew. Math.*, v. 56, 1859, p. 326–344.

2. A. S. WHITE, F. N. COLE, & LOUISE D. CUMMINGS, *Complete Classification of Triad Systems on Fifteen Elements, Memoirs of the National Academy of Sciences*, v. XIV, 1925, Second Memoir, 89 p.

3. C. TOMPKINS, "Machine attacks whose variables are permutations," Amer. Math. Soc. Sixth Symposium on Applied Math. *Proceedings*, McGraw-Hill (in press).

4. R. A. FISHER, "An examination of the different possible solutions of a problem in incomplete blocks," *Annals of Eugenics*, v. 10, 1940, p. 52–75. (Lists 79 systems; the system of Type C, Cole's "Heffter System" was missed.)

# A Set of Test Matrices

**1. Introduction.** For the testing of the adequacy and efficiency of any computing method it is clearly very desirable to have readily available a set of reliable test data. The material discussed below has been found by the author to be of great utility for this purpose. It is surmised that it may also be of interest to many others engaged in the testing of new techniques for the inversion of matrices and the determination of characteristic roots and vectors. Furthermore, the results may be of interest also for purely theoretical reasons.

The exact inverses of a sequence of ten nonsymmetric and again of ten symmetric matrices of extremely poor condition, their determinants, and their largest and smallest characteristic roots and associated vectors as well have all been calculated. These are available in the UMT file. These matrices were obtained by a simple modification of the well known Hilbert matrix, and while they share many features characteristic of that matrix, they differ from it in many other important aspects.

The data calculated also permit an examination of the closeness of certain bounds proposed by OSTROWSKI, PARKER, *et al.*

**2. The sequence of matrices.** The set of matrices mentioned above is obtained from the sequence of "positive" matrices.

$$A_n \equiv (a_{ij}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^{-1} & 3^{-1} & & (n+1)^{-1} \\ 3^{-1} & 4^{-1} & & (n+2)^{-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n^{-1} & (n+1)^{-1} & & (2n-1)^{-1} \end{bmatrix}, \quad n = 1, 2, 3, \cdots.$$

Thus $a_{1j} = 1$ for all $j = 1, 2, 3, \cdots$, and $a_{ij} = (i + j - 1)^{-1}$ for $i = 2, 3, \cdots$; $j = 1, 2, 3, \cdots$.

The matrix $A_n$ has the following properties:

a. The determinant $d_n$ of $A_n$ is of the form $(-1)^{n-1}\delta_n^{-1}$, where $\delta_n > 0$ is an integer.

b. The elements $\alpha_{ij}^{(n)}$ of $A_n^{-1}$ are integers. Further,

$$\sum_{i=1}^{n} \alpha_{ij}^{(n)} = \begin{cases} 1 & \text{for } j = 1 \\ 0 & \text{for } j \neq 1. \end{cases}$$

c. The characteristic root of largest absolute value is positive, that of smallest absolute value negative.

The nonsymmetric matrices $A_n$ are thus finite segments of a matrix $A$ which is closely related to the well known symmetric Hilbert matrix

$$B = (b_{ij}), \qquad b_{ij} = (i + j - 1)^{-1} \tag{2.1}$$

for $i, j = 1, 2, 3, \cdots$. The matrix $B$ has been discussed in many places, e.g., by Todd [1], and many of its properties are now known.

In giving brief sketches in substantiation of some of the properties of $A_n$ we shall prove first that $\delta_n$ satisfies the following recursion formula:

$$\delta_{n+1} = \binom{2n}{n-1}\binom{2n}{n}(2n+1)\delta_n \tag{2.2}$$

for $n = 1, 2, 3, \cdots$, with $\delta_1 = 1$.

Multiplying each column of $d_{n+1}$ by the denominator in the last row, and then dividing outside by the product of these factors, yields

$$d_{n+1} = \frac{1}{(n+1)(n+2)\cdots(2n+1)} \begin{vmatrix} n+1 & n+2 & \cdots & 2n+1 \\ \frac{n+1}{2} & \frac{n+2}{3} & \cdots & \frac{2n+1}{n+2} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

The following further steps, carried out in the order indicated, lead to (2.2):

1. Subtract the last column from each of the others, making use of the fact that

$$\frac{n+j}{i+j-1} - \frac{2n+1}{n+i} = \frac{(n-i+1)(n-j+1)}{(i+j-1)(n+i)}.$$

2. Remove from the first $n$ columns the respective factors $n, n-1, \cdots, 1$ to the outside, and reduce the determinant to the order $n$.

3. Finally remove from all the rows except the first the respective factors

$$\frac{n-1}{n+2}, \frac{n-2}{n+3}, \cdots, \frac{1}{2n},$$

getting

$$\begin{aligned} d_{n+1} &= -\frac{n!(n-1)!}{(n+1)(n+2)\cdots(2n+1)(n+2)(n+3)\cdots 2n} d_n \\ &= -\left[\frac{(n-1)!}{(n+2)\cdots(2n)}\right]^2 \cdot \frac{n}{(2n+1)(n+1)} d_n \\ &= -\binom{2n}{n-1}^{-2} \cdot \frac{n}{(2n+1)(n+1)} d_n, \end{aligned} \tag{2.3}$$

which proves (2.2).

The solution of the difference formula (2.3) leads to

$$d_n = (-1)^{n-1} \frac{[1!2!\cdots(n-2)!(n-1)!]^3}{(n-1)!(n+1)!(n+2)!\cdots(2n-1)!} \tag{2.4}$$

for $n = 1, 2, 3, \cdots$.

Consequently,

$$\det A_n = (-1)^{n-1} \frac{[1!2!\cdots(n-1)!]^4}{1!2!\cdots(2n-1)!} n.$$

In [1] it was shown that for the finite segment $B_n$ of order $n$ of Hilbert's matrix,

$$\det B_n = \frac{[1!2! \cdots (n-1)!]^4}{1!2! \cdots (2n-1)!},$$

so that

$$\det A_n = (-1)^{n-1} n \det B_n.$$

The value of $d_n$ may be estimated quite easily using (2.2) and Stirling's formula $n! \sim (n/e)^n (2\pi n)^{1/2}$; the result is

$$|d_n| \sim 3^n 2^{-2n(n-1)} \phi(n),$$

with $\phi(n)$ of the form $C_0 n^2$, $C_0$ denoting a constant.

TABLE 1a. *Values of* $\delta_n$

| $n$ | |
|---|---|
| 1 | 1 |
| 2 | 6 |
| 3 | 720 |
| 4 | 15 12000 |
| 5 | 5 33433 60000 |
| 6 | 31 05223 67232 00000 |
| 7 | 29541 55782 75110 09280·$10^4$ |
| 8 | 45669 60589 07168 10735·$10^{12}$ |
| 9 | 11430 90871 53174 41088·$10^{22}$ |
| 10 | 46206 89394 79146 91316·$10^{32}$ |

TABLE 1b. *Measures of condition*

| $n$ | $d(A_n)$ | $M(A_n)$ | $P(A_n)$ |
|---|---|---|---|
| 1 | 1 | 1 | |
| 2 | $6^{-1}$ | 12 | 12·587 |
| 3 | $720^{-1}$ | 540 | 354·51 |
| 4 | $(15120 \cdot 10^2)^{-1}$ | 17280 | 13090 |
| 5 | $(53343 \cdot 10^6)^{-1}$ | $67200 \cdot 10^1$ | $45057 \cdot 10^1$ |
| 6 | $(31052 \cdot 10^{12})^{-1}$ | $23814 \cdot 10^3$ | $15259 \cdot 10^3$ |
| 7 | $(29542 \cdot 10^{19})^{-1}$ | $80681 \cdot 10^4$ | $51270 \cdot 10^4$ |
| 8 | $(45670 \cdot 10^{27})^{-1}$ | $28333 \cdot 10^6$ | $17164 \cdot 10^6$ |
| 9 | $(11431 \cdot 10^{37})^{-1}$ | $95447 \cdot 10^7$ | $57364 \cdot 10^7$ |
| 10 | $(46207 \cdot 10^{47})^{-1}$ | $33640 \cdot 10^9$ | $19158 \cdot 10^9$ |

A listing of the values $\delta_n$ for $n = 1, 2, \cdots, 10$, rounded to 20 significant figures, is given in Table 1a. The extremely rapid increase of $\delta_n$ is worthy of note; it serves to underline the poor "condition" of the $A_n$, as measured in terms of $d_n$. As will be shown later other, more suitable measures of condition,

$$M(A) = n \cdot \|A\| \, \|A^{-1}\|, \quad \|A\| = \max_{i,j} |a_{ij}|$$

$$P(A) = |\lambda_{\max}(A)/\lambda_{\min}(A)|,$$

are indicative of the same phenomenon.

Next a few remarks about the inverses $A_n^{-1}$. All the elements of $A_n^{-1}$ are

integers. In fact, it can be shown that for $i = 1, 2, \cdots$, we have

$$\alpha_{i1}^{(n)} = (-1)^{n-i}\binom{n+i-1}{i-1}\binom{n}{i}$$

while for $i = 2, 3, \cdots$ and $j = 1, 2, \cdots, n$, we have

$$\alpha_{i,j+1}^{(n)} = (-1)^{i-j}k_{i,j+1}\binom{n+i-1}{i+j}\binom{n+j}{i+j}$$

where

$$k_{i,j+1} = \binom{i+j}{j}\binom{i+j-1}{j-1}i.$$

Let us proceed to the characteristic roots $\lambda_i$ and characteristic vectors $x_i$ of $A_n$, arranging the $\lambda_i$ according to increasing absolute value: $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n|$; we shall set $\lambda_1 \equiv \lambda_m$, $\lambda_n \equiv \lambda_M$. Let the vector $x_i$ have the components $x_{ij}$, $j = 1, 2, \cdots, n$.

Since

$$\det[A - \lambda I] = (-1)^n\lambda[\lambda^{n-1} - (\Sigma_i a_{ii})\lambda^{n-2} + \cdots] + (-1)^{n-1}\delta_n^{-1},$$

it follows immediately that there is always at least one positive characteristic root. It seems that there is only one positive characteristic root, the others being negative.

The existence of a positive characteristic root also follows from the result of FROBENIUS [2] that the root $\lambda_M$ of largest absolute value of any positive non-decomposable matrix is positive and simple, and may be associated with a proper vector having positive components only, the only vector possible that possesses this property.

Estimates for the $\lambda_i$ are readily calculated from the following theorem due to PARKER [3]:

*Let $U = (u_{ij})$ be an arbitrary matrix of order $n$. Put*

$$\xi = n^{-1}\sum_{i=1}^{n}|u_{ii}|; \quad P_i = \sum_{j=1,\, j\neq i}^{n}|u_{ij}|, \quad Q_j = \sum_{i=1,\, i\neq j}^{n}|u_{ij}|;$$

$$S_i = P_i + |u_{ii} - \xi|, \quad S = \max_i S_i; \quad T_j = Q_j + |u_{jj} - \xi|; \quad T = \max_j T_j.$$

*Then for every characteristic root $\lambda$ of $U$*

$$|\lambda - \xi| \leq \min(S, T). \tag{2.5}$$

*Furthermore, the number $\xi$ may be replaced by any other number.*

Applying (2.5) to the matrix $A_n$, for $\xi = 0$, it is seen that the maximum column sum $T$ is a bound:

$$\lambda_M^{(n)} < \log(n+1) + \gamma, \tag{2.6}$$

with $\gamma = 0.577\cdots$ denoting Euler's constant.

Bounds for the components of $x_M$ may be obtained by means of Ostrowski's

theorem [4]:

*Let $A_n = (a_{ij})$ be a matrix of order n of positive elements. Put*

$$R_i = \sum_{j=i}^{n} a_{ij}, \quad i = 1, 2, \cdots, n; \qquad R = \max_i R_i, \quad r = \min_i R_i:$$

$$k_1 = \min_i a_{ii}, \quad k_2 = \min_{i \neq j} a_{ij}; \qquad \sigma_1 = [(r - k_1)/(R - k_1)]^{1/2}.$$

*Further, let the positive components $x_{Mj}$ of the characteristic vector $x_M$ belonging to $\lambda_M$ be normalized so that*

$$x_{Mn} \leq x_{M,n-1} \leq \cdots \leq x_{M2} \leq x_{M1} = 1.$$

*Then*

$$k_2/(R - r + k_2) < x_{Mn} \leq \sigma_1$$

*provided $R \neq r$.*

In our case

$$R = n, \quad k_1 = (2n - 1)^{-1}, \quad k_2 = (2n - 2)^{-1}, \quad r = \sum_{j=1}^{n} (n + j - 1)^{-1}.$$

It is found that

$$(2.7) \qquad (2n^2)^{-1}[1 + O(n^{-1})] < x_{Mn} < [\log 2/n]^{1/2}[1 + O(n^{-1})].$$

The characteristic roots of least absolute value $\lambda_m$ are frequently also of interest. If the $\lambda_i^{(n)}$ are distinct, then

$$A_n^{-1} = \sum_{i=1}^{n} (1/\lambda_i^{(n)}) x_i^{(n)} \cdot r_i^{(n)}$$

where the $x_i^{(n)}$ denote the characteristic column vectors of $A_n$ belonging to $\lambda_i^{(n)}$, and the $r_i^{(n)}$ are the characteristic row vectors of the transposed matrix $A_n^T$ belonging to $\lambda_i^{(n)}$. The smallest root $\lambda_m^{(n)}$ may then be found as the dominant root $\mu_M^{(n)}$ of $A_n^{-1}$.

If it is further known that the roots $\lambda$ of the matrix $U = (u_{ij})$ are *real*, then (2.5) permits the conclusion

$$\lambda \geq \xi - \min(S, T);$$

if, in addition, the particular root $\lambda = \mu_M$ is negative, then clearly

$$(2.8) \qquad \mu_M^{-1} \leq [\xi - \min(S, T)]^{-1}.$$

Another upper bound for $\lambda_m$ is provided by the fact that for *definite* matrices

$$(n\|A\|)^{-1} \leq \|A^{-1}\| \leq |\lambda_m|^{-1}.$$

Thus

$$(2.9) \qquad |\lambda_m| \leq \|A^{-1}\|^{-1}.$$

**3. The inverses and determinants.** The first ten inversions were carried out by the method of partitioning [5]. A tabulation of the inverse $A_6^{-1}$ computed by this method is given in Table 2.

TABLE 2. *The inverse of* $A_6$

$$A_6^{-1} = \begin{bmatrix} -6 & 630 & -6720 & 22680 & -30240 & 13860 \\ 105 & -7350 & 88200 & -3\ 17520 & 4\ 41000 & -2\ 07900 \\ -560 & 29400 & -3\ 76320 & 14\ 11200 & -20\ 16000 & 9\ 70200 \\ 1260 & -52920 & 7\ 05600 & -27\ 21600 & 39\ 69000 & -19\ 40400 \\ -1260 & 44100 & -6\ 04800 & 23\ 81400 & -35\ 28000 & 17\ 46360 \\ 462 & -13860 & 1\ 94040 & -7\ 76160 & 11\ 64240 & -5\ 82120 \end{bmatrix}$$

Certain interesting properties of these inverses have already been pointed out before. The following features of the $A_n^{-1}$ are also noteworthy:

a. It is seen that

$$\operatorname{sgn} \alpha_{ij}^{(n)} = \begin{cases} (-1)^{i+j-1} \text{ for } i \geq 1,\ j \geq 1, \text{ for } n \text{ even} \\ (-1)^{i+j-1} \text{ for } i \geq 1,\ j \geq 2, \text{ for } n \text{ odd} \\ (-1)^{i+j} \quad \text{for } i \geq 1,\ j = 1, \text{ for } n \text{ odd.} \end{cases}$$

b. $\|A_n^{-1}\| = \alpha_{k+1,k+1}^{(n)}$ or $\alpha_{k,k+1}^{(n)}$ with $k+1 < n$ for $n > 3$.

Now

$$\|A_n^{-1}\| \geq |\alpha_{nn}^{(n)}| \sim C_1 2^{4n}.$$

Inspection of the $\|A_n^{-1}\|$ suggests that

$$\|A_n^{-1}\| \sim C_2 2^{4n} \quad \text{with} \quad C_2 \approx 4 \cdot 10^{-3}. \tag{3.1}$$

With these estimates it is seen that the measures $M(A_n)$ are approximately

$$M(A_n) \sim C_2 n 2^{4n}, \tag{3.2}$$

so that the order of ill-conditioning is about that of Hilbert's matrix.

The exact values of the $M(A_n)$ are shown in Table 1b.

It has been stated [6] that "average" matrices have $M$-condition numbers of the order $n^{1/2} \log n$, and $P$-condition numbers of the order $n$. In the light of such orders for condition numbers the matrices $A_n$ must indeed be called extremely pathological.

There is a large number of techniques in matrix algebra which are specifically restricted to symmetric positive matrices $H$; for the use of such procedures there have been calculated the matrices

$$H_n = A_n^T A_n.$$

From the reciprocals of $A_n$ those of $H_n$ are easily calculated as

$$H_n^{-1} = A_n^{-1}(A_n^{-1})^T. \tag{3.3}$$

The inverses $H_n^{-1}$ thus also consist of integer elements only. It is known (TAUSSKY [8]) that the condition (in several senses) of $AA^T$ is worse than that of $A$.

Obviously in $H_{k+1} = (h_{ij}^{(k+1)})$,

$$\begin{aligned} h_{ij}^{(k+1)} &= 1 + [(i+1)(j+1)]^{-1} + [(i+2)(j+2)]^{-1} + \cdots \\ &\qquad + [(i+k)(j+k)]^{-1} \\ &= h_{ij}^{(k)} + [(i+k)(j+k)]^{-1} \end{aligned} \tag{3.4}$$

for all $i, j \leq k$, $k = 1, 2, \cdots$, with $h_{ij}^{(1)} = 1$. Consequently,

$$h_{ij}^{(k+1)} = 1 + \Psi'(i) - \sum_{j=k+1}^{\infty} (i+j)^{-2},$$

where $\Psi(x) = \Gamma'(x+1)/\Gamma(x+1)$. Further, for fixed $k$,

$$\max_{i,j} h_{ij}^{(k)} = h_{11}^{(k)} = 1 + \Psi'(1) - \sum_{j=k}^{\infty} (1+j)^{-2} = \pi^2/6 - \sum_{j=k}^{\infty} (i+j)^{-2}. \tag{3.5}$$

A listing of $H_6$ is provided in Table 3.

TABLE 3. *The matrix* $H_6$

$$H_6 = \begin{bmatrix} \frac{5369}{3600} & \frac{19}{14} & \frac{431}{336} & \frac{1867}{1512} & \frac{12107}{10080} & 1\ \frac{62997}{1\ 38600} \\ & 2\ \frac{22581}{1\ 76400} & \frac{29}{24} & \frac{169}{144} & \frac{1241}{1080} & \frac{17911}{15840} \\ & & 8\ \frac{22949}{7\ 05600} & \frac{41}{36} & \frac{403}{360} & \frac{6563}{5940} \\ & & & 70\ \frac{88041}{63\ 50400} & \frac{11}{10} & \frac{359}{330} \\ & & & & 68\ \frac{97529}{63\ 50400} & \frac{71}{66} \\ & & & & & 8196\ \frac{07009}{7683\ 98400} \end{bmatrix}$$

**4. Characteristic roots and vectors.** Since there is a unique characteristic root $\lambda_M^{(n)}$ of largest absolute value the power method [7] may be used to isolate $\lambda_M^{(n)}$, while simultaneously obtaining the associated characteristic vector $x_M^{(n)}$:

$$\begin{aligned} &A_n x_{k-1}^{(n)} = \lambda_k^{(n)} x_k^{(n)}, \quad k = 1, 2, 3, \cdots \\ &\lim_{k\to\infty} \lambda_k^{(n)} = \lambda_M^{(n)}, \quad \lim_{k\to\infty} x_k^{(n)} = x_M^{(n)}, \end{aligned} \tag{4.1}$$

TABLE 4. *Characteristic roots*

| $n$ | $\lambda_M^{(n)}$ | $\lambda_m^{(n)}$ |
|---|---|---|
| 1 | 1.000 000 | |
| 2 | 1.448 403 | −.115 0693 |
| 3 | 1.707 105 | $-.481\ 5399 \cdot 10^{-2}$ |
| 4 | 1.886 632 | $-.144\ 1324 \cdot 10^{-2}$ |
| 5 | 2.022 999 | $-.448\ 9833 \cdot 10^{-3}$ |
| 6 | 2.132 376 | $-.139\ 7499 \cdot 10^{-3}$ |
| 7 | 2.223 362 | $-.433\ 6577 \cdot 10^{-4}$ |
| 8 | 2.301 055 | $-.134\ 0623 \cdot 10^{-4}$ |
| 9 | 2.368 717 | $-.412\ 9309 \cdot 10^{-11}$ |
| 10 | 2.428 554 | $-.126\ 7649 \cdot 10^{-12}$ |

TABLE 5a. *Characteristic vectors* $x_M^{(n)}$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 | 1.000 000 |
| | | .448 403 | .416 793 | .394 224 | .376 917 | .363 036 | .351 545 | .341 809 | .333 408 | .326 051 |
| | | | .290 313 | .277 320 | .266 965 | .258 417 | .251 183 | .244 943 | .239 479 | .234 636 |
| | | | | .215 088 | .208 099 | .202 206 | .197 135 | .192 700 | .188 772 | .185 257 |
| | | | | | .171 019 | .166 679 | .162 893 | .159 546 | .156 554 | .153 855 |
| | | | | | | .142 038 | .139 091 | .136 461 | .134 092 | .131 940 |
| | | | | | | | .121 514 | .119 387 | .117 457 | .115 694 |
| | | | | | | | | .106 209 | .104 603 | .103 129 |
| | | | | | | | | | .094 352 | .093 099 |
| | | | | | | | | | | .084 893 |

TABLE 5b. *Characteristic vectors* $x_m^{(n)}$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| | | − .896 805 | .550 163 | − .224 188 | .081 470 | − .027 443 | .008 795 | − .002 721 | .000 819 | − .000 242 |
| | | 1.000 000 | −1.552 812 | 1.270 271 | − .772 313 | .392 118 | − .176 722 | .073 168 | − .028 430 | .010 516 |
| | | | 1.000 000 | −2.046 050 | 2.237 419 | −1.768 987 | 1.140 790 | − .638 751 | .322 286 | − .150 100 |
| | | | | 1.000 000 | −2.546 576 | 3.450 596 | −3.339 147 | 2.598 579 | −1.732 159 | 1.027 660 |
| | | | | | 1.000 000 | −3.046 283 | 4.912 246 | −5.607 667 | 5.099 020 | − 3.934 520 |
| | | | | | | 1.000 000 | −3.545 962 | 6.623 060 | −8.699 495 | 9.038 123 |
| | | | | | | | 1.000 000 | −4.045 669 | 8.583 374 | −12.739 602 |
| | | | | | | | | 1.000 000 | −4.545 416 | 10.793 366 |
| | | | | | | | | | 1.000 000 | − 5.045 200 |
| | | | | | | | | | | 1.000 000 |

provided the initial arbitrary vector $x_0^{(n)}$ is not orthogonal to $x_M^{(n)}$. The rate of convergence increases with the ratio between $\lambda_M^{(n)}$ and the next characteristic root of $A_n$.

The absolutely largest elements of $A_n$ occur in the first row; it is therefore advisable to normalize the $x_M^{(n)}$ by keeping its first element equal to unity. In carrying out the computation a considerable saving in labor may be further achieved by utilizing the values of $x_M^{(n)}$ obtained at the $n$th step to start the next step with a vector $x_0^{(n+1)}$ whose first $n$ components were identical with $x_M^{(n)}$. Proceeding in this manner there was obtained in seven or less iterations an agreement to six decimal places in both characteristic roots and vectors.

The characteristic roots $\lambda_M^{(n)}$ are shown in the middle column of Table 4; as indicated by (2.6) they actually do not grow faster than $\log n$.

The characteristic vectors $x_M^{(n)}$ belonging to the $\lambda_M^{(n)}$ are listed in Table 5a. Their smallest components $x_{M_n}^{(n)}$ seem to decrease as $n^{-1}$. The bounds given in (2.7) thus seem a bit wide.

The characteristic roots $\lambda_m^{(n)} = \mu_M^{(n)}$ of least absolute value were also calculated, together with their characteristic vectors $x_m^{(n)}$, now normalized to have their last component equal to unity. The values of $\lambda_m^{(n)}$ are shown in the third column of Table 4. Here the inequality (2.8), with $\xi = 0$, is found to restrict the roots fairly well, the upper bounds—$\min (S^{(n)}, T^{(n)})^{-1}$ being, at least for $n \leq 10$, less than 30 per cent too high.

For purposes of comparison the resulting condition numbers $P(A_n)$ were also computed; they are exhibited in the last column of Table 1b. These suggest the following rough estimate:

$$P(A_n) \sim C_3 2^{3n} \log n, \quad C_3 \approx 8 \cdot 10^{-3}, \tag{4.2}$$

which, with (3.2), gives,

$$P(A_n) \sim (2/n) \log n M(A_n); \tag{4.3}$$

Todd [1] found for Hilbert's matrix $B$

$$\pi n^{-1} M(B_n) < P(B_n) < \pi M(B_n).$$

MARK LOTKIN

AVCO Manufacturing Corporation
Stratford, Connecticut

1. JOHN TODD, "The condition of the finite segments of the Hilbert Matrix," in *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues*, NBS Applied Math. Series No. 39, 1954, p. 109–116.
2. G. FROBENIUS, "Über Matrizen aus positiven Elementen," *Sitzungsberichte Kgl. Preuss. Akad. Wiss.*, 1908, p. 471–476; 1909, p. 514–518.
3. W. V. PARKER, "Characteristic roots and the field of values of a matrix," *Duke Math. Jn.*, v. 15, 1948, p. 439–442.
4. A. OSTROWSKI, "Bounds for the greatest latent root of a positive matrix," London Math. Soc., *Jn.*, v. 27, 1952, p. 253–256.
5. MARK LOTKIN & RUSSELL REMAGE, "Scaling and error analysis for matrix inversion by partitioning," *Ann. Math. Stat.*, 24, 1953, p. 428–439.
6. JOHN TODD, "The condition of a certain matrix," Cambridge Phil. Soc., *Proc.*, v. 46, 1949, p. 116–118.
7. R. A. FRAZER, W. J. DUNCAN, & A. R. COLLAR, *Elementary Matrices*, Cambridge Univ. Press, 1950.
8. OLGA TAUSSKY, "Note on the condition of matrices," *MTAC*, v. 4, 1940, p. 111–112.

# Some Comments on a NORC Computation of $\pi$

Among the problems suggested as demonstration routines for the NORC [1] was the calculation of $\pi$ to a large number of digits. In 1949, $\pi$ was calculated to 2035D on the ENIAC [2]. The present computation on the NORC, which was carried out by the authors at the Watson Scientific Computing Laboratory, produced 3089D. This limit was chosen since the entire calculation could be contained in the NORC's high speed memory (2000 locations). The program was designed to produce any number of digits up to this limit. More extensive programming involving use of the magnetic tapes would have increased the number of digits by several orders of magnitude.

The formula used was the same as that employed for the ENIAC computation:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$

in conjunction with the GREGORY series

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1}.$$

The form of the expression used for the NORC computation was:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{100(0.2)^{2n+3} - (1/239)^{2n+1}}{2n+1} \right].$$

All intermediate factors were truncated without rounding to 3093D. For $n = 0$, the value of $100(0.2)^{2n+3}$ was set at 0.8; the value of this factor at each succeeding $n$ was computed by multiplying the value at $n - 1$ by .04, with truncation of the product to 3093D. For $n = 0$, the value of $(1/239)^{2n+1}$ was computed with truncation of the quotient to 3093D; the value of this factor at each succeeding $n$ was computed by dividing the value at $n - 1$ by $239^2$, with truncation of the quotient to 3093D. For each $n$, the tens complement of this second factor was taken and added to the first factor; the sum was divided by $2n + 1$, with truncation to 3093D. This result or its tens complement, according to the value of $n$, was added to $\sum_{0}^{n-1}$. This process was continued until the first 3093D of the second factor were zeros; computations involving this factor were then bypassed. The process terminated when the first 3093D of the first factor were zeros. A count was kept of the number of terms required for the calculation; at termination, $n$ equaled 2214.

A second run, with the same program, of the calculation was made about two months after the first run. A comparison of the two results showed agreement. The first 2035D of the ENIAC and NORC results agree. Since the programming for the computation on the NORC is completely general for any number of digits, the correctness of the digits up to 3089D is strongly indicated. Further assurance of the correctness of the result is afforded by the NORC's inherent checking system [3]. Note that the only expected error results from truncation of the intermediate calculations; this error is a function of $n$ and could affect the last

NORC Computation of π

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
82148 08651 32823 06647 09384 46095 50582 23172 53594 08128
48111 74502 84102 70193 85211 05559 64462 29489 54930 38196
44288 10975 66593 34461 28475 64823 37867 83165 27120 19091
45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436
78925 90360 01133 05305 48820 46652 13841 46951 94151 16094
33057 27036 57595 91953 09218 61173 81932 61179 31051 18548
07446 23799 62749 56735 18857 52724 89122 79381 83011 94912
98336 73362 44065 66430 86021 39494 63952 24737 19070 21798
60943 70277 05392 17176 29317 67523 84674 81846 76694 05132
00056 81271 45263 56082 77857 71342 75778 96091 73637 17872
14684 40901 22495 34301 46549 58537 10507 92279 68925 89235
42019 95611 21290 21960 86403 44181 59813 62977 47713 09960
51870 72113 49999 99837 29780 49951 05973 17328 16096 31859
50244 59455 34690 83026 42522 30825 33446 85035 26193 11881
71010 00313 78387 52886 58753 32083 81420 61717 76691 47303
59825 34904 28755 46873 11595 62863 88235 37875 93751 95778
18577 80532 17122 68066 13001 92787 66111 95909 21642 01989
38095 25720 10654 85863 27886 59361 53381 82796 82303 01952
03530 18529 68995 77362 25994 13891 24972 17752 83479 13151
55748 57242 45415 06959 50829 53311 68617 27855 88907 50983
81754 63746 49393 19255 06040 09277 01671 13900 98488 24012
85836 16035 63707 66010 47101 81942 95559 61989 46767 83744
94482 55379 77472 68471 04047 53464 62080 46684 25906 94912
93313 67702 89891 52104 75216 20569 66024 05803 81501 93511
25338 24300 35587 64024 74964 73263 91419 92726 04269 92279
67823 54781 63600 93417 21641 21992 45863 15030 28618 29745
55706 74983 85054 94588 58692 69956 90927 21079 75093 02955
32116 53449 87202 75596 02364 80665 49911 98818 34797 75356
63698 07426 54252 78625 51818 41757 46728 90977 77279 38000
81647 06001 61452 49192 17321 72147 72350 14144 19735 68548
16136 11573 52552 13347 57418 49468 43852 33239 07394 14333
45477 62416 86251 89835 69485 56209 92192 22184 27255 02542
56887 67179 04946 01653 46680 49886 27232 79178 60857 84383
82796 79766 81454 10095 38837 86360 95068 00642 25125 20511
73929 84896 08412 84886 26945 60424 19652 85022 21066 11863
06744 27862 20391 94945 04712 37137 86960 95636 43719 17287
46776 46575 73962 41389 08658 32645 99581 33904 78027 59009
94657 64078 95126 94683 98352 59570 98258 22620 52248 94077
26719 47826 84826 01476 99090 26401 36394 43745 53050 68203
49625 24517 49399 65143 14298 09190 65925 09372 21696 46151
57098 58387 41059 78859 59772 97549 89301 61753 92846 81382
68683 86894 27741 55991 85592 52459 53959 43104 99725 24680
84598 72736 44695 84865 38367 36222 62609 91246 08051 24388
43904 51244 13654 97627 80797 71569 14359 97700 12961 60894
41694 86855 58484 06353 42207 22258 28488 64815 84560 28506
01684 27394 52267 46767 88952 52138 52254 99546 66727 82398
64565 96116 35488 62305 77456 49803 55936 34568 17432 41125
15076 06947 94510 96596 09402 52288 79710 89314 56691 36867
22874 89405 60101 50330 86179 28680 92087 47609 17824 93858
90097 14909 67598 52613 65549 78189 31297 84821 68299 89487
22658 80485 75640 14270 47755 51323 79641 45152 37462 34364
54285 84447 95265 86782 10511 41354 73573 95231 13427 16610
21359 69536 23144 29524 84937 18711 01457 65403 59027 99344
03742 00731 05785 39062 19838 74478 08478 48968 33214 45713
86875 19435 06430 21845 31910 48481 00537 06146 80674 91927
81911 97939 95206 14196 63428 75444 06437 45123 71819 21799
98391 01591 95618 14675 14269 12397 48940 90718 64942 31961
56794 52080 95146 55022 52316 03881 93014 20937 62137 85595
66389 37787 08303 90697 92077 34672 21825 6259

four digits of the final result. The result has been tabulated to 3089D; the final digit is unrounded.

Running time for the 3093D was approximately thirteen minutes. The programming takes account of the number of zeros generated to the right of the decimal point in each factor, so that the number of operations required for each term in the series decreases. This leads to the following statement—if the time to compute $\pi$ to $m$ digits is $t$ units, then the time to produce $km$ digits is roughly $k^2t$ units; this holds true as long as the calculation is contained in high-speed storage.

The following table gives a count of each of the digits in $\pi$.

| (1) | (2) 1–3090 | (3) 1–2036 | (4) 2037–3090 | (5) (4)/(3) |
|---|---|---|---|---|
| 0 | 269 | 184 | 85 | .46 |
| 1 | 315 | 213 | 102 | .47 |
| 2 | 314 | 210 | 104 | .50 |
| 3 | 276 | 191 | 85 | .45 |
| 4 | 322 | 198 | 124 | .63 |
| 5 | 326 | 211 | 115 | .54 |
| 6 | 311 | 204 | 107 | .52 |
| 7 | 297 | 200 | 97 | .49 |
| 8 | 318 | 207 | 111 | .54 |
| 9 | 342 | 218 | 124 | .57 |
| $\Sigma$ | 3090 | 2036 | 1054 | .52 |

S. C. NICHOLSON
J. JEENEL

Watson Scientific Computing Laboratory
612 West 115th Street
New York 27, New York

1. The IBM-Naval Ordnance Research Calculator, now located at Naval Proving Ground, Dahlgren, Virginia.
2. GEORGE W. REITWIESNER, "An ENIAC determination of $\pi$ and $e$ to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11–15.
3. For a description of the NORC checking system, see W. J. ECKERT & R. B. JONES, *Faster, Faster*, McGraw-Hill Book Company, New York, 1955, p. 98–104.

# Orthogonal Polynomials Arising in the Numerical Evaluation of Inverse Laplace Transforms

**Abstract.** In finding $f(t)$, the inverse LAPLACE transform of $F(p)$, where (1) $f(t) = (1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^{pt}F(p)dp$, the function $F(p)$ may be either known only numerically or too complicated for evaluating $f(t)$ by CAUCHY's theorem. When $F(p)$ behaves like a polynomial without a constant term, in the variable $1/p$, along $(c - j\infty, c + j\infty)$, one may find $f(t)$ numerically using new quadrature formulas (analogous to those employing the zeros of the LAGUERRE polynomials in the direct Laplace transform). Suitable choice of $p_i$ yields an $n$-point quadrature

formula that is exact when $\rho_{2n}$ is any arbitrary polynomial of the $(2n)$th degree in $x \equiv 1/p$ without a constant term, namely: (2) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p \rho_{2n}(1/p)dp = \sum_{i=1}^{n} A_i^{(n)} \rho_{2n}(1/p_i)$. In (2), $x_i \equiv 1/p_i$ are the zeros of the orthogonal polynomials $p_n(x) \equiv \prod_{i=1}^{n} (x - x_i)$ where (3) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p (1/p)p_n(1/p)(1/p)^i dp = 0$, $i = 0, 1, \cdots, n-1$ and $A_i^{(n)}$ correspond to the CHRISTOFFEL numbers. The normalization $P_n(1/p) \equiv (4n-2)(4n-6)\cdots 6p_n(1/p)$, $n \geq 2$, produces all integral coefficients. $P_n(1/p)$ is proven to be $(-1)^n e^{-p} p^n d^n(e^p/p^n)/dp^n$. The normalization factor is proved, in three different ways, to be given by (4) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty} e^p (1/p)[P_n(1/p)]^2 dp = \frac{1}{2}(-1)^n$. Proofs are given for the recurrence formula (5) $(2n-3)P_n(x) = [(4n-2)(2n-3)x+2]P_{n-1}(x) + (2n-1)P_{n-2}(x)$, for $n \geq 3$, and the differential equation (6) $x^2 P_n''(x) + (x-1)P_n'(x) - n^2 P_n(x) = 0$. The quantities $p_i^{(n)}$, $1/p_i^{(n)}$ and $A_i^{(n)}$ were computed, mostly to 6S – 8S, for $i = 1(1)n$, $n = 1(1)8$.

**I. Introduction: Occurrence of inverse Laplace transforms.** For a given function of $p$, $F(p)$, which is the direct Laplace transform of some unknown function $f(t)$, for $t > 0$, one usually finds the $f(t)$ from the following explicit expression:

$$(1) \qquad f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{pt} F(p)dp.$$

Formula (1) is known as the inverse Laplace transform of $F(p)$. In (1) the quantity $c$ is a real constant $\pm 0$ that is greater than the real part of all the singular points of $F(p)$. In practice $c$ is usually positive, but $c$ can be negative as long as for $f(t)$ satisfying Dirichlet's conditions in any finite positive interval the integral $\int_0^\infty e^{-ct} f(t)dt$ is absolutely convergent (H. S. CARSLAW and J. C. JAEGER [1]).

A note by the referee follows this paper and indicates relations between the present work and work published elsewhere.

The examples treated in most textbooks on operational calculus and Laplace transforms contain such functions $F(p)$ that their poles and branch points (and residues also) are obtainable without too much difficulty, and the inversion integral in (1) is evaluated by suitable deformation of the path of integration, and the use of Cauchy's theorem. But there are countless other examples where $F(p)$ might be too complicated to yield explicit information about the location and nature of its singularities without a prohibitive amount of labor. For instance, one will recall that in most textbook examples treating the solution of ordinary and partial differential equations by operational means, the original system of differential equations is transformed into a system whose solution $F(p)$ is usually some known elementary function or a very extensively tabulated function of a simple differential equation (like a Bessel function), so that its analytic character and singularities are well known. But in actual practice one might not be fortunate enough to obtain such a comparatively simple $F(p)$. Thus the transformed differential equations might not yield a known function. Instead it might be amenable

only to solution in series or numerical integration, with an $F(p)$ that is given as a tabulated function of $p$. Also, the solution $F(p)$ might be given explicitly in closed form as a combination of integrals of such complicated analytic expressions that it might be easier to evaluate it for different numerical values of $p$ than to find its poles, residues, and branch points.

The purpose of this present article is to discuss the properties of a new set of orthogonal polynomials which can be the basis for convenient formulas for approximating $f(t)$ in (1) for different positive values of $t$ when one has an $F(p)$ that is too complicated to show its analytic character, but which can be calculated for any $p$.

All further discussion will now be for $F(p)$ assumed to be exactly of the form $\sum_{r=1}^{n} \frac{a_r}{p^r}$, i.e., a polynomial in $1/p$ without a constant term.

To obtain a definite integral without a parameter $t$ in the exponential term, which is the "weight function," let $pt = u$ in (1), so that we obtain

$$f(t) = \frac{1}{2\pi jt} \int_{c_1-j\infty}^{c_1+j\infty} e^u F\left(\frac{u}{t}\right) du, \tag{1'}$$

where $F\left(\frac{u}{t}\right)$ is still a polynomial in $1/u$, without a constant term.

**II. Use of orthogonal polynomials.** At this point one may recall the application of the theory of orthogonal polynomials to quadrature formulas for definite integrals where the integrand is the product of a preassigned weight function and a polynomial $P(t)$. There it is possible to employ the value of $P(t)$ at $n$ fixed irregularly spaced points $t_i$, $i = 1, 2, \cdots, n$, such that the resulting quadrature formula is exact when $P(t)$ is any arbitrary polynomial of $(2n-1)$-th degree. Thus for the direct Laplace transform of $P(t)$, namely $\int_0^\infty e^{-pt}P(t)dt$, which is essentially $\int_0^\infty e^{-t}Q(t)dt$ for polynomial $Q(t)$, the points $t_i$ are taken equal to the zeros of the Laguerre polynomials, which have been tabulated extensively (H. E. SALZER and R. ZUCKER [2]). In the present case, even though we are not dealing with a polynomial in $p$, we can still solve the problem of finding a Gaussian-type quadrature formula for (1') of approximately double the degree of accuracy of an ordinary quadrature formula based upon the same number of equally spaced points.

Thus let $\rho_{2n}(1/p)$ be any arbitrary $(2n)$-th degree polynomial in the variable $1/p$, which vanishes at $1/p = 0$. Consider $n$ distinct points $1/p_i$, $i = 1, 2, \cdots, n$ other than $1/p = 0$ and construct the $(n+1)$-point Lagrangian polynomial approximation (of the $n^{\text{th}}$ degree in $1/p$), to $\rho_{2n}(1/p)$, based upon the points $1/p_i$, $i = 1, 2, \cdots, n$ and $1/p = 0$. The $(n+1)$th point $1/p = 0$ is needed in order to provide for the property that $\rho_{2n}(1/p)$ vanishes at $p = \infty$. We have for this polynomial approximation $L^{(n+1)}(1/p)$ the explicit expression

$$L^{(n+1)}\left(\frac{1}{p}\right) = \sum_{i=1}^{n+1} L_i^{(n+1)}\left(\frac{1}{p}\right) \rho_{2n}\left(\frac{1}{p_i}\right), \tag{2}$$

where on the right hand side of (2)

$$(3) \qquad L_i^{(n+1)}\left(\frac{1}{p}\right) \equiv \prod_{k=1}^{n+1}{}'\left(\frac{1}{p}-\frac{1}{p_k}\right)\Big/\prod_{k=1}^{n+1}{}'\left(\frac{1}{p_i}-\frac{1}{p_k}\right),$$

the $\prod'$ denoting the absence of $k = i$. In (2), $p_{n+1}$ is $\infty$, so that there is actually no $(n+1)$-th term in (2) and $L_{n+1}^{(n+1)}(1/p)$ is not used. (The summation in (2) is written with $n+1$ instead of $n$ to avoid confusion with the $(n-1)$-th degree coefficients $L_i^{(n)}(1/p)$ which differ from the $L_i^{(n+1)}(1/p)$ by not having the factor $p_i/p$.)

Following the method in G. SZEGÖ [3], we consider the $(2n)$-th degree polynomial in $1/p$, namely, $\rho_{2n}(1/p) - L^{(n+1)}(1/p)$ which vanishes at $1/p = 0, 1/p_i$, $i = 1, 2, \cdots, n$, and thus has

$$\frac{1}{p}p_n\left(\frac{1}{p}\right) \equiv \frac{1}{p}\prod_{i=1}^{n}\left(\frac{1}{p}-\frac{1}{p_i}\right)$$

as a factor. Writing

$$(4) \qquad \rho_{2n}\left(\frac{1}{p}\right) = L^{(n+1)}\left(\frac{1}{p}\right) + \frac{1}{p}p_n\left(\frac{1}{p}\right)r_{n-1}\left(\frac{1}{p}\right),$$

it follows that

$$(5) \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p\rho_{2n}\left(\frac{1}{p}\right)dp = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p L^{(n+1)}\left(\frac{1}{p}\right)dp + \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p\frac{1}{p}p_n\left(\frac{1}{p}\right)r_{n-1}\left(\frac{1}{p}\right)dp.$$

Thus if the second term in the right member of (5) always vanishes, (5) will be an $n$-point quadrature formula that is exact for any $(2n)$-th degree polynomial in $1/p$ without a constant term, namely,

$$(6) \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p\rho_{2n}\left(\frac{1}{p}\right)dp = \sum_{i=1}^{n} A_i^{(n)}\rho_{2n}\left(\frac{1}{p_i}\right),$$

where the "Christoffel numbers" $A_i^{(n)}$ are given by [6]

$$(7) \qquad A_i^{(n)} \equiv \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p L_i^{(n+1)}\left(\frac{1}{p}\right)dp.$$

A sufficient condition for (6) to hold is obviously the "orthogonality" of $(1/p)p_n(1/p)$ with respect to any arbitrary $\rho_{n-1}(1/p)$, namely,

$$(8) \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p\frac{1}{p}p_n\left(\frac{1}{p}\right)\left(\frac{1}{p}\right)^i dp = 0, \quad i = 0, 1, \cdots, n-1.$$

The necessity of (8) is also obvious from (6) by choosing

$$\rho_{2n}(1/p) = (1/p)p_n(1/p)\rho_{n-1}(1/p)$$

where $\rho_{n-1}(1/p)$ is any arbitrary polynomial in $1/p$ of the $(n-1)$-th degree.

Hence the points $1/p_i$, now denoted by $1/p_i^{(n)}$, are the zeros of a certain set of orthogonal polynomials in the variable $1/p$.

The condition of orthogonality (8) is also mathematically equivalent, in terms of actual polynomials (by setting $x = 1/p$), to having a polynomial of the $n^{\text{th}}$ degree $q_n(x)$ which is orthogonal to any $\rho_{n-1}(x)$, with weight function $e^{1/x}/x$, where the path of integration is a circle of radius $1/2c$ whose center is at $(1/2c, 0)$.

If the polynomial $p_n(1/p)$ is written as

$$\left(\frac{1}{p}\right)^n + b_{n-1}\left(\frac{1}{p}\right)^{n-1} + b_{n-2}\left(\frac{1}{p}\right)^{n-2} + \cdots + b_1\left(\frac{1}{p}\right) + b_0,$$

the determination of $b_i$, $i = 0, 1, \cdots, n-1$, to satisfy the conditions of orthogonality (8), making use of

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^p}{p^{m+1}}\,dp = \frac{1}{m!}$$

is in the solution of this system of linear equations:

$$(9')\quad \begin{cases} \dfrac{1}{n!} + \dfrac{b_{n-1}}{(n-1)!} + \dfrac{b_{n-2}}{(n-2)!} + \cdots + \dfrac{b_1}{1!} + \dfrac{b_0}{0!} = 0 \\ \dfrac{1}{(n+1)!} + \dfrac{b_{n-1}}{n!} + \dfrac{b_{n-2}}{(n-1)!} + \cdots + \dfrac{b_1}{2!} + \dfrac{b_0}{1!} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{(2n-1)!} + \dfrac{b_{n-1}}{(2n-2)!} + \dfrac{b_{n-2}}{(2n-3)!} + \cdots + \dfrac{b_1}{n!} + \dfrac{b_0}{(n-1)!} = 0. \end{cases}$$

For numerical work it is somewhat easier to solve (9′) in the form

$$(9)\quad \begin{cases} 1 + nb_{n-1} + n(n-1)b_{n-2} + \cdots + n!b_1 + n!b_0 = 0 \\ 1 + (n+1)b_{n-1} + (n+1)nb_{n-2} + \cdots \\ \qquad + (n+1)\cdots 3b_1 + (n+1)\cdots 2b_0 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 + (2n-1)b_{n-1} + (2n-1)(2n-2)b_{n-2} + \cdots \\ \qquad + (2n-1)\cdots(n+1)b_1 + (2n-1)\cdots nb_0 = 0. \end{cases}$$

**III. Explicit expression for orthogonal polynomials.** It is convenient to normalize the polynomials $p_n(x)$, where $x \equiv 1/p$, by multiplying $p_n(x)$, for $n \geq 2$, by $(4n-2)(4n-6)\cdots 6$. This normalization produces polynomials with all coefficients integral (proven below) and it is not the usual normalization by multiplication by

$$\left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left\{p_n\left(\frac{1}{p}\right)\right\}^2 dp\right]^{-1}.$$

Denoting $(4n-2)(4n-6)\cdots 6p_n(1/p)$ by $P_n(1/p)$ for $n \geq 2$, and $p_1(1/p)$ by $P_1(1/p)$, one can avoid the labor of solving (9′) or (9) directly by showing that

$P_n(1/p)$ has the following more elegant definition:

$$P_n\left(\frac{1}{p}\right) = (-1)^n e^{-p} p^n \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right). \tag{10}$$

That (10) yields the leading coefficient of $1/p^n$ in $P_n(1/p)$, namely

$$\begin{cases} 1, & \text{for } n = 1, \\ (4n-2)(4n-6)\cdots 6, & \text{for } n \geq 2, \end{cases}$$

is obvious by induction. To prove the orthogonality property, or (8), it suffices to prove the vanishing of

$$\text{(A)} \quad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[(-1)^n e^{-p} p^n \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\right]\left[(-1)^m e^{-p} p^m \frac{d^m}{dp^m}\left(\frac{e^p}{p^m}\right)\right] dp$$

for $m < n$. This last expression is written as

$$\text{(B)} \quad \frac{(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{-p} p^{m+n-1} \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\frac{d^m}{dp^m}\left(\frac{e^p}{p^m}\right) dp,$$

and after integrating by parts $m$ times, noting that the integrated parts always vanish, we have

$$\text{(C)} \quad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{d^m}{dp^m}\left[e^{-p} p^{m+n-1} \frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right)\right]\frac{e^p}{p^m} dp,$$

which by LEIBNITZ's rule is expressible as

$$\text{(D)} \quad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \sum_{r=0}^{m}\binom{m}{r}\frac{d^r}{dp^r}(e^{-p} p^{m+n-1})\frac{d^{m-r+n}}{dp^{m-r+n}}\left(\frac{e^p}{p^n}\right)\frac{e^p}{p^m} dp.$$

Application of Leibnitz's rule a second time to

$$\frac{d^r}{dp^r}(e^{-p} p^{m+n-1})$$

in the above and cancellation of $e^p/p^m$, yields

$$\text{(E)} \quad \frac{(-1)^m(-1)^{m+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty} \sum_{r=0}^{m}\binom{m}{r}\left[\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\binom{m+n-1}{s} s!\, p^{n-1-s}\right] \times \frac{d^{m-r+n}}{dp^{m-r+n}}\left(\frac{e^p}{p^n}\right) dp.$$

Now we integrate by parts $(m - r + n)$ times each term of the above double summation. The integrated part will always vanish since it will have a factor of $1/p$ to at least the first power. Furthermore, at some stage in the partial integration of each term, that stage varying with the term, the integral part will also vanish if $m < n$. This last follows because the lowest power of $p^{n-1-s}$ is positive or zero, since $s$ can equal at the most $r$ which can equal at the most $m \leq n - 1$. Then in the integration by parts the positive or zero power $p^{n-1-s}$, for each value

of $s$ between 0 and $r$, would always be eventually annulled because the initially occurring differential operator $d^{m-r+n}/dp^{m-r+n}$ is of order $m - r + n > n - 1 - s$ even for the highest value of $n - 1 - s$ when $s = 0$ (due to $m - r + n > n - 1$ for every $r$ between 0 and $m$). Thus (E) vanishes, which proves (10), and establishes at the same time that this normalization yields all integral coefficients for $P_n(1/p)$.

**IV. Normalization factor.** To obtain the normalization factor, which turns out to be given by

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[P_n\left(\frac{1}{p}\right)\right]^2 dp = \tfrac{1}{2}(-1)^n, \tag{11}$$

we repeat the preceding argument for $m = n$ and now notice that in the final integral (E) the lowest power of $p^{n-1-s}$ will survive the integration by parts, because it is equal to $1/p$. Retaining in the double summation in (E) only the single non-vanishing term $s = r = m = n$, we get

$$\frac{(-1)^{2n+n}}{2\pi j}\int_{c-j\infty}^{c+j\infty}\binom{2n-1}{n} n!\,\frac{1}{p}\frac{d^n}{dp^n}\left(\frac{e^p}{p^n}\right) dp, \tag{F}$$

which is integrated by parts $n$ times, the integrated part always vanishing, to give

$$\frac{(-1)^{2n+2n}}{2\pi j}\binom{2n-1}{n} n!\int_{c-j\infty}^{c+j\infty} e^p \frac{(-1)^n\cdot 2\cdot 3\cdots(n-1)n}{p^{2n+1}}\,dp. \tag{G}$$

But (G) is

$$(-1)^n\binom{2n-1}{n} n!n!\frac{1}{(2n)!} = \frac{(-1)^n(2n-1)(2n-2)\cdots n}{n!}\,n!n!\frac{1}{(2n)!}$$

which reduces to $(-1)^n n/2n$ or $\frac{1}{2}(-1)^n$, thus proving (11).

From (10), the explicit formula for $P_n(1/p)$ is seen to be [7]

$$\begin{aligned} P_n\left(\frac{1}{p}\right) = (-1)^n\Bigg[&\frac{(-1)^n\binom{2n-1}{n}n!}{p^n} \\ &+\frac{(-1)^{n-1}\binom{n}{1}\binom{2n-2}{n-1}(n-1)!}{p^{n-1}} \\ &+\frac{(-1)^{n-2}\binom{n}{2}\binom{2n-3}{n-2}(n-2)!}{p^{n-2}}+\cdots \\ &+\frac{(-1)^{n-r}\binom{n}{r}\binom{2n-r-1}{n-r}(n-r)!}{p^{n-r}}+\cdots \\ &+\frac{n^2(-1)^1}{p}+(-1)^0\Bigg]. \end{aligned} \tag{12}$$

**V. Recurrence formula.** It is easy to obtain the recurrence relation for the polynomials $P_n(x)$ by employing a fundamental theorem about the existence of a recurrence formula connecting any three successive orthogonal polynomials (G. Szegö [8]), namely,

$$P_n(x) = (a_n x + b_n)P_{n-1}(x) + c_n P_{n-2}(x). \tag{13}$$

Thus $a_n$ is immediately seen to be $4n - 2$. Equating constant terms in (13), one finds $c_n = b_n + 1$, and after substitution into the equation derived from the coefficients of $x$, one obtains

$$b_n = \frac{2}{2n-3}, \quad c_n = \frac{2n-1}{2n-3},$$

so that the recurrence formula satisfied by $P_n(x)$ is seen to be [9]

$$(2n-3)P_n(x) = [(4n-2)(2n-3)x + 2]P_{n-1}(x) + (2n-1)P_{n-2}(x), \quad \text{for } n \geq 3. \tag{14}$$

From (14) and (8) only, without making use of (10), one can again find the normalization factor given in (11), through the following inductive argument:

Multiply (14) by $P_{n-2}(x)$ and then operate with

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} \cdots dp$$

to obtain (making use of (8)):

$$0 = \frac{(4n-2)(2n-3)}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p} P_{n-1}\left(\frac{1}{p}\right)\cdot\frac{1}{p} P_{n-2}\left(\frac{1}{p}\right) dp + 0 + \frac{(2n-1)}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^p \frac{1}{p}\left[P_{n-2}\left(\frac{1}{p}\right)\right]^2 dp.$$

Denoting the left member of (11) by $F_n$, still making use of (8) to replace in the first of the above integrals $(1/p)P_{n-2}(1/p)$ by

$$\frac{1}{a_{n-1}} P_{n-1}\left(\frac{1}{p}\right) = \frac{1}{4n-6} P_{n-1}\left(\frac{1}{p}\right),$$

one now obtains

$$0 = \frac{(4n-2)(2n-3)}{4n-6} F_{n-1} + (2n-1)F_{n-2},$$

or $F_{n-1} = -F_{n-2}$. Since $F_1 = -\frac{1}{2}$, (11) follows by induction.

The normalization given in (11) can be seen in a third way, directly from the

explicit formula for $P_n(1/p)$ in (12). For, in view of (8), it suffices to consider only

$$\text{(H)} \qquad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{e^p}{p}\frac{(-1)^{2n}}{p^n}\binom{2n-1}{n}n!P_n\left(\frac{1}{p}\right)dp,$$

or

$$\text{(I)} \qquad (-1)^{2n}\binom{2n-1}{n}n!\sum_{r=0}^{n}\frac{(-1)^{2n-r}\binom{n}{r}\binom{2n-r-1}{n-r}(n-r)!}{(2n-r)!},$$

or

$$\text{(J)} \qquad \binom{2n-1}{n}n!\sum_{r=0}^{n}(-1)^r\frac{n(n-1)\cdots(n-r+1)}{r!}\,\frac{(2n-r-1)\cdots(n+1)n}{(2n-r)(2n-r-1)\cdots(n+1)n!},$$

which, after cancellations, is written as

$$\text{(K)} \qquad \sum_{r=0}^{n}(-1)^r\frac{(2n-1)(2n-2)\cdots n}{(n-r)!n!}\,\frac{n}{2n-r}\,\frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!},$$

or

$$\text{(L)} \qquad n\sum_{r=0}^{n}(-1)^r\frac{1}{2n-r}\,\frac{(2n-1)(2n-2)\cdots n}{(n-r)!r!},$$

and this, in turn, is expressible in the form

$$\text{(M)} \quad \tfrac{1}{2}(-1)^n \times\sum_{r=0}^{n}\frac{(2n-0)(2n-1)(2n-2)\cdots(2n-[r-1])(2n-[r+1])\cdots(2n-n)}{(r-0)(r-1)\cdots(r-[r-1])(r-[r+1])\cdots(r-n)}.$$

In $(M)$, the $\frac{1}{2}(-1)^n$ is multiplied by the sum of the coefficients of the Lagrangian interpolation polynomial for the $(n+1)$ points $0, 1, \cdots, n$, for the variable equal to $2n$. But that sum is identically equal to 1, i.e., for any value, $2n$ or otherwise. Thus we obtain once more $\frac{1}{2}(-1)^n$ for the normalization.

**VI. Integral coefficients.** It may be of interest to show that (14) alone, without any knowledge of (10), implies that $P_n(x)$ has integral coefficients. We prove this by noting that $P_{n+1}(x)$ will have integral coefficients if $P_m(x)$, $m \leq n$, has integral coefficients and the following identical polynomial congruence holds for $m = n+1$:

$$(15) \qquad 2P_{m-1}(x) + (2m-1)P_{m-2}(x) \equiv 0 \pmod{(2m-3)}.$$

Now the existence of integral coefficients of $P_m(x)$ and congruence (15) can be verified for the first few values of $m$. We then show that if (15) holds for some particular $m = n$, it holds for $m = n+1$, provided $P_m(x)$, $m \leq n-1$, has

integral coefficients, or, in other words, that

$$2P_n(x) + (2n+1)P_{n-1}(x) \equiv 0 \pmod{(2n-1)}.$$

This last congruence, by (14), is equivalent to

$$\frac{4}{2n-3}P_{n-1}(x) + \frac{2(2n-1)}{2n-3}P_{n-2}(x) + (2n+1)P_{n-1}(x) \equiv 0 \pmod{(2n-1)},$$

or to

$$\frac{(2n-1)^2}{2n-3}P_{n-1}(x) + \frac{2(2n-1)}{2n-3}P_{n-2}(x) \equiv 0 \pmod{(2n-1)},$$

which in turn is expressible as

$$(2n-1)\left[\frac{(2n-1)P_{n-1}(x) + 2P_{n-2}(x)}{2n-3}\right] \equiv 0 \pmod{(2n-1)},$$

or

$$(2n-1)\left[\frac{(2+(2n-3)P_{n-1}(x) + (2n-1-(2n-3))P_{n-2}(x)}{2n-3}\right] \equiv 0 \pmod{(2n-1)}.$$

But under the assumptions that (15) holds for $m = n$, and that $P_m(x), m \leq n-1$, has integral coefficients, the last quantity in brackets is a polynomial with integral coefficients, which shows that the last congruence is satisfied identically in $x$. Thus (15) holds for $m = n+1$ and $P_{n+1}(x)$ has integral coefficients. We proceed in this way to every $n$. There is a slight subtlety in the argument of this induction in the sense that the integral coefficients of $P_m(x)$ up to $m = n-1$ only are needed to go from $m = n$ to $m = n+1$ in (15), but then use is made of the integral coefficients of $P_n(x)$ in using (14) with $n+1$ in place of $n$.

**VII. Differential equation.** It is easy to show that $P_n(x)$ satisfies the differential equation

$$x^2P_n''(x) + (x-1)P_n'(x) - n^2P_n(x) = 0. \tag{16}$$

Thus one merely expresses (12) in the form

$$P_n(x) = (-1)^n\left[1 + \sum_{r=1}^{n}\frac{(-1)^r n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-\overline{r-1}^2)x^r}{r!}\right], \tag{12'}$$

and then observes that (12′) is equivalent to the automatically terminating "infinite series."

$$P_n(x) = \sum_{r=0}^{\infty} a_r x^r, \tag{12''}$$

where

$$a_0 = (-1)^n, \quad \text{and} \quad ra_r = -(n^2 - \overline{r-1}^2)a_{r-1}, \quad \text{for} \quad r > 0.$$

Working backwards from (12″), by equating coefficients of $x^{r-1}$, one sees that (12″) must arise from (16).

**VIII. Explicit expressions for polynomials.** Because these polynomials $P_n(x)$ are of fundamental importance, and their role in the inverse Laplace transform is comparable to the role of the Laguerre polynomials in the direct Laplace transform, their explicit expressions are given below for $n = 1(1)12$:

$$P_1(x) = x - 1$$

$$P_2(x) = 6x^2 - 4x + 1$$

$$P_3(x) = 60x^3 - 36x^2 + 9x - 1$$

$$P_4(x) = 840x^4 - 480x^3 + 120x^2 - 16x + 1$$

$$P_5(x) = 15120x^5 - 8400x^4 + 2100x^3 - 300x^2 + 25x - 1$$

$$P_6(x) = 3\ 32640x^6 - 1\ 81440x^5 + 45360x^4 - 6720x^3 + 630x^2 - 36x + 1$$

$$P_7(x) = 86\ 48640x^7 - 46\ 56960x^6 + 11\ 64240x^5 - 1\ 76400x^4 + 17640x^3 - 1176x^2 + 49x - 1$$

$$P_8(x) = 2594\ 59200x^8 - 1383\ 78240x^7 + 345\ 94560x^6 - 53\ 22240x^5 + 5\ 54400x^4 - 40320x^3 + 2016x^2 - 64x + 1$$

$$P_9(x) = 88216\ 12800x^9 - 46702\ 65600x^8 + 11675\ 66400x^7 - 1816\ 21440x^6 + 194\ 59440x^5 - 14\ 96880x^4 + 83160x^3 - 3240x^2 + 81x - 1$$

$$P_{10}(x) = 33\ 52212\ 86400x^{10} - 17\ 64322\ 56000x^9 + 4\ 41080\ 64000x^8 - 69189\ 12000x^7 + 7567\ 56000x^6 - 605\ 40480x^5 + 36\ 03600x^4 - 1\ 58400x^3 + 4950x^2 - 100x + 1$$

$$P_{11}(x) = 1407\ 92940\ 28800x^{11} - 737\ 48683\ 00800x^{10} + 184\ 37170\ 75200x^9 - 29\ 11132\ 22400x^8 + 3\ 23459\ 13600x^7 - 26637\ 81120x^6 + 1664\ 86320x^5 - 79\ 27920x^4 + 2\ 83140x^3 - 7260x^2 + 121x - 1$$

$$P_{12}(x) = 64764\ 75253\ 24800x^{12} - 33790\ 30566\ 91200x^{11} + 8447\ 57641\ 72800x^{10} - 1340\ 88514\ 56000x^9 + 150\ 84957\ 88800x^8 - 12\ 70312\ 24320x^7 + 82335\ 05280x^6 - 4151\ 34720x^5 + 162\ 16200x^4 - 4\ 80480x^3 + 10296x^2 - 144x + 1.$$

**IX. Zeros and Christoffel numbers.** In the numerical table below there are given the values of the reciprocals of the zeros of $P_n(x)$ or $p_i^{(n)}$, the zeros of $P_n(x)$, or $1/p_i^{(n)}$, and the corresponding Christoffel numbers $A_i^{(n)}$, for $n = 1(1)8$. Use of these quantities in the quadrature formula (6) above can give theoretically exact accuracy for any polynomial in $1/p$ (with no constant term) up to the 16th degree. However, the fact that these tabulated values of $p_i^{(n)}$, $1/p_i^{(n)}$ and $A_i^{(n)}$ are correct to only about a unit in the last significant figure that is given, must

TABLE OF ZEROS AND CHRISTOFFEL NUMBERS

| | $p_i^{(n)}$ | $1/p_i^{(n)}$ | $A_i^{(n)}$ |
|---|---|---|---|
| | $n = 1$ | | |
| $i = 1$: | 1.00000 000 + .00000 000$i$ | 1.00000 0000 +.00000 0000$i$ | 1.00000 000 + .00000 000$i$ |
| | $n = 2$ | | |
| $i = 1$: | 2.00000 000 − 1.41421 356$i$ | .33333 3333 +.23570 2260$i$ | − 1.00000 00 − 3.53553 39$i$ |
| $i = 2$: | 2.00000 000 + 1.41421 356$i$ | .33333 3333 −.23570 2260$i$ | − 1.00000 00 + 3.53553 39$i$ |
| | $n = 3$ | | |
| $i = 1$: | 2.68108 288 − 3.05043 020$i$ | .16255 5585 +.18494 9324$i$ | − 7.64874 9 + 4.17164 0$i$ |
| $i = 2$: | 2.68108 288 + 3.05043 020$i$ | .16255 5585 −.18494 9324$i$ | − 7.64874 9 − 4.17164 0$i$ |
| $i = 3$: | 3.63783 425 + .00000 000$i$ | .27488 8830 +.00000 0000$i$ | 18.29749 8 + .00000 0$i$ |
| | $n = 4$ | | |
| $i = 1$: | 3.21280 69 − 4.77308 75$i$ | .09705 0482 +.14418 2470$i$ | 11.3015 + 12.4717$i$ |
| $i = 2$: | 3.21280 69 + 4.77308 75$i$ | .09705 0482 −.14418 2470$i$ | 11.3015 − 12.4717$i$ |
| $i = 3$: | 4.78719 30 − 1.56747 65$i$ | .18866 3804 +.06177 4421$i$ | − 13.30154 − 60.07173$i$ |
| $i = 4$: | 4.78719 30 + 1.56747 65$i$ | .18866 3804 −.06177 4421$i$ | − 13.30154 + 60.07173$i$ |
| | $n = 5$ | | |
| $i = 1$: | 3.65569 43 − 6.54373 69$i$ | .06506 5779 +.11646 8528$i$ | 15.8268 − 24.1256$i$ |
| $i = 2$: | 3.65569 43 + 6.54373 69$i$ | .06506 5779 −.11646 8528$i$ | 15.8268 + 24.1256$i$ |
| $i = 3$: | 5.70095 33 − 3.21026 56$i$ | .13317 9077 +.07499 4511$i$ | − 149.9984 + 68.0423$i$ |
| $i = 4$: | 5.70095 33 + 3.21026 56$i$ | .13317 9077 −.07499 4511$i$ | − 149.9984 − 68.0423$i$ |
| $i = 5$: | 6.28670 47 + .00000 00$i$ | .15906 5845 +.00000 0000$i$ | 273.3433 + .0000$i$ |

TABLE OF ZEROS AND CHRISTOFFEL NUMBERS—*Continued*

| | $p_i^{(n)}$ | | $1/p_i^{(n)}$ | | $A_i^{(n)}$ | |
|---|---|---|---|---|---|---|
| | | | $n = 6$ | | | |
| $i = 1$: | 4.03884 | 8 − 8.34560 0$i$ | .04698 447 | +.09708 551$i$ | − 43.516 | − 14.015$i$ |
| $i = 2$: | 4.03884 | 8 + 8.34560 0$i$ | .04698 447 | −.09708 551$i$ | − 43.516 | + 14.015$i$ |
| $i = 3$: | 6.47051 | 3 − 4.90012 1$i$ | .09821 855 | +.07438 093$i$ | 226.060 | + 305.103$i$ |
| $i = 4$: | 6.47051 | 3 + 4.90012 1$i$ | .09821 855 | −.07438 093$i$ | 226.060 | − 305.103$i$ |
| $i = 5$: | 7.49064 | 0 − 1.62149 9$i$ | .12752 426 | +.02760 517$i$ | − 185.544 | − 917.794$i$ |
| $i = 6$: | 7.49064 | 0 + 1.62149 9$i$ | .12752 426 | −.02760 517$i$ | − 185.544 | + 917.794$i$ |
| | | | $n = 7$ | | | |
| $i = 1$: | 4.37869 | 4 −10.16969 3$i$ | .03571 656 | +.08295 315$i$ | − 2.053 | + 68.442$i$ |
| $i = 2$: | 4.37869 | 4 +10.16969 3$i$ | .03571 656 | −.08295 315$i$ | − 2.053 | − 68.442$i$ |
| $i = 3$: | 7.14105 | 5 − 6.62304 6$i$ | .07528 041 | +.06981 960$i$ | 515.229 | − 595.468$i$ |
| $i = 4$: | 7.14105 | 5 + 6.62304 6$i$ | .07528 041 | −.06981 960$i$ | 515.229 | + 595.468$i$ |
| $i = 5$: | 8.51183 | 5 − 3.28101 4$i$ | .10228 557 | +.03942 750$i$ | −2490.669 | + 1040.334$i$ |
| $i = 6$: | 8.51183 | 5 + 3.28101 4$i$ | .10228 557 | −.03942 750$i$ | −2490.669 | − 1040.334$i$ |
| $i = 7$: | 8.93683 | 3 + .00000 0$i$ | .11189 646 | +.00000 000$i$ | 3961.985 | + .000$i$ |
| | | | $n = 8$ | | | |
| $i = 1$: | 4.68549 | 5 −12.01057 8$i$ | .02819 058 | +.07226 240$i$ | 94.94 | − 25.48$i$ |
| $i = 2$: | 4.68549 | 5 +12.01057 8$i$ | .02819 058 | −.07226 240$i$ | 94.94 | + 25.48$i$ |
| $i = 3$: | 7.73869 | 0 − 8.37088 1$i$ | .05954 718 | +.06441 172$i$ | −1334.9 | − 702. 4$i$ |
| $i = 4$: | 7.73869 | 0 + 8.37088 1$i$ | .05954 718 | −.06441 172$i$ | −1334.9 | + 702. 4$i$ |
| $i = 5$: | 9.40637 | 0 − 4.96922 0$i$ | .08311 501 | +.04390 820$i$ | 3848.5 | + 5690. 3$i$ |
| $i = 6$: | 9.40637 | 0 + 4.96922 0$i$ | .08311 501 | −.04390 820$i$ | 3848.5 | − 5690. 3$i$ |
| $i = 7$: | 10.16944 | 4 − 1.64920 3$i$ | .09581 390 | +.01553 837$i$ | −2613. | −13549. $i$ |
| $i = 8$: | 10.16944 | 4 + 1.64920 3$i$ | .09581 390 | −.01553 837$i$ | −2613. | +13549. $i$ |

be taken into account in any practical example where some upper bound for the error should be estimated.

HERBERT E. SALZER

Ordnance Corps
Diamond Ordnance Fuze Laboratories
Washington, D. C.

NOTE BY REFEREE

The function $F(p)$ is subject to certain restrictions because it is a Laplace transform. In order for $F(p)$ to be the Laplace transform of the function $f(t)$ given by (1), it is sufficient that $F(p)$ have the form (cf. G. DOETSCH [5]):

$$F(p) = a/p + F_1(p)/p^{1+\delta},$$

where $\delta > 0$, $a$ is a constant, and $F_1(p)$ is analytic and bounded in the half plane $\mathrm{Re}(p) > c$. We assume that this condition is satisfied. Whether this condition is also sufficient for the convergence of the $n$-point quadrature formula to the true value of $f(t)$ in (1), when $n$ tends to infinity, has not been determined. The author makes use here of the fact that the convergence occurs whenever $F(p)$ is a polynomial in $1/p$ without a constant term; in fact, the quadrature is exact for polynomials of degree not greater than $2n$. G. SZEGÖ [10] has shown that under quite general conditions a Gauss-Jacobi type quadrature formula which converges for polynomials also converges for a much wider class of functions. Unfortunately his theorems do not seem to apply directly to the present case because the integral (1) involves a complex valued weight function which is not of bounded variation.

1. H. S. CARSLAW & J. C. JAEGER, *Operational Methods in Applied Mathematics*, 2nd edition, Oxford University Press, 1949, p. 75.
2. H. E. SALZER & R. ZUCKER, "Table of the Zeros and Weight Factors of the First Fifteen Laguerre Polynomials," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 1004–1012.
3. G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc., *Colloquium Pub.*, v. 23, 1939, p. 46-47.
4. H. L. KRALL & O. FRINK, "A New Class of Orthogonal Polynomials: The Bessel Polynomials," Amer. Math. Soc., *Trans.*, v. 65, 1, 1949, p. 100–115.
5. G. DOETSCH, *Theorie und Anwendung der Laplace-Transformation*, Springer, Berlin, 1937, p. 128.
6. The shift in notation from $(n+1)$ to $n$ in $A_i^{(n)}$ will cause no confusion after the $A_i$'s have been computed and are ready for use in (6).
7. It was called to the author's attention by H. L. KRALL that $P_n(x) = (-1)^n y_n(x, 1, -1)$ where $y_n(x, a, b)$ are "generalized Bessel polynomials" (see [4]).
8. G. SZEGÖ, *op. cit.*, p. 41–42.
9. Formula (14) holds for $n = 2$ if we define $P_0(x) = 1$.
10. G. SZEGÖ, *op. cit.*, p. 341–342.

# On the Improvement of the Solutions to a Set of Simultaneous Linear Equations using the ILLIAC

The basic method used for solving simultaneous linear equations on the University of Illinois' electronic digital computer, the ILLIAC, has already been described in detail by WHEELER and NASH [1]. The routine currently in use on the ILLIAC, programmed by Wheeler [2], makes use of the method of elimination to solve the set of $n$ simultaneous linear equations

$$\sum_{j=0}^{n-1} a_{ij}x_j + a_{in} = 0 \quad i = 0, 1, 2, \cdots, n-1 \tag{1}$$

in a manner very similar to that used by a human solving such a system.

In brief, the procedure used is as follows:

a) The augmented matrix

$$a_{ij} \qquad \begin{matrix} i = 0, 1, 2, \cdots, n-1 \\ j = 0, 1, 2, \cdots, n \end{matrix} \tag{2}$$

is scaled (by reducing each row by an appropriate factor) and punched as decimals with up to 12 places, row by row, on the input tape; each row is followed by an identifying symbol.

b) The computer reads this matrix a row at a time, converts the numbers to binary form, and eliminates an additional variable from each additional row by using the previous rows which are stored in the memory of the machine. When this process is completed the matrix (2) is stored in ILLIAC as the triangular matrix

$$(3) \qquad c_{ij} \qquad \begin{matrix} i = 0, 1, 2, \cdots, n-1 \\ j = 0, 1, 2, \cdots, n \end{matrix}$$

with $c_{ij} = 0$ for $0 \leq j < i \leq n-1$.

c) The solutions are obtained by a process of back substitution. Due to the fact that ILLIAC can represent numbers only in the range $-1$ to 1, provision must be made for automatic scaling of the results. If bars indicate the ILLIAC representation of a number $(-1 \leq \bar{x} < 1)$, and if $\bar{x}_n = 10^{-p}$ represents a scale factor, then the scaled solutions are formed during the back substitution by successively applying

$$(4) \qquad \bar{x}_i = -\sum_{j=i+1}^{n} \frac{c_{ij}\bar{x}_j}{c_{ii}} \quad i = n-1, n-2, \cdots, 1, 0.$$

The true solutions are given in terms of the scaled solutions by

$$(5) \qquad x_i = \frac{\bar{x}_i}{\bar{x}_n} = \frac{\bar{x}_i}{10^{-p}} \quad i = 0, 1, 2, \cdots, n-1.$$

The program first attempts to carry out the back substitution with $p = 1$. If at any stage the sum in (4) exceeds scale, $p$ is increased by unity and the process is repeated. After the back substitution has been successfully completed the solutions $\bar{x}_i$ are printed in columnar form followed by $\bar{x}_n$ which serves to locate the decimal point in the column. Thus the machine actually operates on the set of equations

$$(6) \qquad \sum_{j=0}^{n-1} a_{ij}\bar{x}_j + a_{in}\bar{x}_n = 0 \quad i = 0, 1, 2, \cdots, n-1$$

whose solutions can be forced to remain within scale by proper adjustment of $\bar{x}_n$.

This method is extremely straightforward and is in fact quite similar to that which would be used to solve such a set of equations "by hand". The chief utility of such a process applied to an automatic computing machine is that it minimizes the space necessary to store the matrix of coefficients in the equations. Any method that demands storage of the entire matrix $a_{ij}$ requires $n(n+1) \sim n^2$ memory locations. This elimination method uses the input tape as auxiliary storage and forms the reduced triangular matrix $c_{ij}$ as this tape is read in and thus requires only $\frac{1}{2}n(n+3) \sim \frac{1}{2}n^2$ memory locations.

If the set of equations being solved is poorly conditioned in having a nearly

singular determinant,

$$\text{Det } (a_{ij}) \text{ small} \quad i, j = 0, 1, 2, \cdots, n-1, \tag{7}$$

the above outlined process can yield solutions which are in error compared to the true solutions. Such errors can be aggravated by rounding-off in the many arithmetical steps of the process and by the truncation errors due to the finite register length of 39 binary digits. In order to obtain an improved set of solutions in such circumstances an iterative application of the method of residues has been programmed [3] for the ILLIAC. If $x'_i$ indicate the solutions to the equations (1) obtained by the ILLIAC and if corrections $\epsilon_i$ are defined by

$$x_i \equiv x'_i + \epsilon_i \tag{8}$$

where the $x_i$ are the actual solutions of (1), then the corrections can be found from

$$\sum_{j=0}^{n-1} a_{ij}\epsilon_j + \delta_i = 0 \quad i = 0, 1, 2, \cdots, n-1 \tag{9}$$

where $\delta_i$ are the residues defined by

$$\delta_i \equiv \sum_{j=0}^{n-1} a_{ij}x'_j + a_{in} \quad i = 0, 1, 2, \cdots, n-1. \tag{10}$$

Just as the original solutions were in error due to the finite register length and the rounding-off errors in the arithmetic processes so are the $\epsilon_i$ in error. It could not be expected that a single application of equations (9) to find the corrections $\epsilon_i$ would yield the correct solutions to (1) by the use of (8).

Hence, it has proved useful to apply the method of residues several times and also artificially to increase the numerical size of a residue contained in the machine by multiplying the residue by a factor $B = 2^s$ in order to gain numerical accuracy in the solution of (9). Let superscripts zero designate the solutions obtained by the conventional solution of the equation (1) as outlined in the above equations (3)–(5). Then if $x_i^{(k)}$ represent the solutions at the $k^{\text{th}}$ stage of the iterative process, the residues at this stage are defined by:

$$\delta_i^{(k)}x_n^{(0)} \equiv \sum_{j=0}^{n-1} a_{ij}x_j^{(k)} + a_{in}x_n^{(0)} \quad i = 0, 1, 2, \cdots, n-1, \tag{11}$$

where ILLIAC representations of the numbers are implied even though the bars have been dropped. The scale factor $x_n^{(0)}$ as determined by the first solution of equations (6) is maintained throughout the process. The corrections $\epsilon_i^{(k)}$ at the $k^{\text{th}}$ stage are determined by dividing the result of (11) by $x_n^{(0)}$ to form $\delta_i$ and solving the equations:

$$\sum_{j=0}^{n-1} a_{ij}x_j^{(k)} + B\delta_i^{(k)} = 0 \quad i = 0, 1, 2, \cdots, n-1. \tag{12}$$

The sum in (11) is formed by double precision continued multiplication which is a facility readily available on the ILLIAC. Hence the $\delta_i^{(k)}$ are capable of accepting a multiplication by the scaling up factor $B$ and still retain a register of significance.

The equations (12) are identical in form to equations (1) and can be solved in the same fashion. The $(k+1)^{\text{th}}$ approximation to the solutions are:

$$x_i^{(k+1)} = x_i^{(k)} + B^{-1}\epsilon_i^{(k)} \frac{x_n^{(0)}}{x_n^{(k)}} \quad i = 0, 1, 2, \cdots, n-1 \tag{13}$$

where $x^{(k)}$ is the automatic scale factor supplied by ILLIAC in the solution of (12) which guarantees that each iterate is expressed in the same scale.

To use Library Routine 100 the scaled coefficients of the augmented matrix are placed upon the input tape just as described in the use of Library Routine 51. To produce the $x_i^{(0)}$ the internal operation of the routine is identical to steps (a)–(c) outlined above. On each subsequent reading of the input tape the routine computes the residues (11) as each row of the input tape is read, solves the equations (12), forms the improved solutions (13), and prints them. At the $k^{\text{th}}$ stage the factor $B$ is set to be

$$B = 2^{6+4k} \quad k = 1, 2, \cdots. \tag{14}$$

If at any stage, multiplication by such a factor would cause the expression on the left hand side of equation (11) to exceed scale, the machine stops, replaces $B$ by $B' = B2^{-2}$ and puts itself into such a condition that the reading of the coefficient tape for the $k^{\text{th}}$ stage can be repeated. If the equations are so badly conditioned that equation (11) will accept multiplication by no $B > 1$ without exceeding scale, this program is to no avail; the machine stops after producing a characteristic indicating signal. If the problem has been removed from the machine after the $k^{\text{th}}$ approximations to the solutions have been obtained it is unnecessary to repeat the process from the beginning if at a later time further iterations are desired. Provisions have been made so that the program, after being placed in ILLIAC, will first read in the $x_i^{(k)}$ from some previous iteration and store them in such a way that the condition of the machine is the same as if the iterative process had been continuing and had reached the $k^{\text{th}}$ stage. Subsequent iterations can hence be resumed from this stage.

This method of annihilation of residues was chosen and used in ILLIAC to minimize storage requirements. By making the arithmetic procedures during subsequent iterations as nearly like those used on the first solution, the length of the operating routine stored in the memory has been reduced since most of it serves double duty. Again, only a triangular matrix need be stored at each iteration stage. This economy of memory space has however been achieved by the sacrifice of some speed since the tape of coefficients must be read on each iteration and the triangular matrix must be reformed at each iteration. The total tape reading time is at most $\frac{1}{30}kn^2$ sec. where $k$ is the number of iterations necessary; the total computation time is $\frac{1}{300}kn^3$ sec., which is essentially dominated by the $n^3$ multiplications necessary to form the reduced triangular matrix. This program was designed to use only high speed storage so that even with the increased efficiency of storage yielded by this procedure $n$ is still limited to moderate values so that the above times are not excessive in spite of the $k$ repetitions.

This method would be undesirable if enough storage were available in order to allow $n$ to become large (order of 100, say) especially if that storage were rela-

tively slow (for example, a drum). In such a case the entire matrix should be read and stored or processed but once (reading time proportional to $n^2$ not $kn^2$) and some iteration scheme requiring only $n^2$ multiplications per iteration used (computation time proportional to $kn^2$ not $kn^3$). This gain in computation speed is not quite so favorable as it may appear due to the greatly increased access time to the slower memory.

The ILLIAC's memory is sufficient to accommodate a system of 39 equations when used with Routine 51. The additional length of Routine 100 restricts to 37 the number of equations that it can handle. With 37 equations the operation time of Routine 100 is about 4 minutes per iteration. For fewer equations the operation time is very roughly proportional to the square of the number of equations.

The utility of this residue annihilating routine may be illustrated by an example comparing the results obtained with Routines 51 and 100 for sets of equations purposely constructed to be poorly condition to various degrees.

The set of equations:

$$(15) \qquad (1 - 10^{-q}) \quad \begin{aligned} x_1 - x_2 + x_3 - x_4 + 3 &= 0 \\ x_1 - x_2 + x_3 - x_4 + 2 &= 0 \\ x_1 - x_2 \qquad\quad + 1 &= 0 \\ x_1 \qquad\quad - x_4 + 3 &= 0 \end{aligned}$$

has been constructed to have

$$(16) \qquad \begin{aligned} |\mathrm{Det}\,(a_{ij})\,1 &= 10^{-q} \\ x_1 &= 10^q \\ x_2 &= 10^q + 1 \\ x_3 &= 10^q + 2 \\ x_4 &= 10^q + 3. \end{aligned}$$

These equations were solved to eleven decimal digits for $1 \leq q \leq 9$ by both Routines 51 and 100 as indicated in parentheses with the results:

$q = 1$

| $x_i$(51) | $x_i^{(0)}$(100) | $x_i^{(1)}$(100) |
|---|---|---|
| 10.000 000 05 | 10.000 000 004 | 10.000 000 000 |
| 11.000 000 02 | 11.000 000 000 | 11.000 000 000 |
| 12.000 000 06 | 12.000 000 004 | 12.000 000 000 |
| 13.000 000 04 | 13.000 000 002 | 13.000 000 000 |

$q = 2$

| $x_i$(51) | $x_i^{(0)}$(100) | $x_i^{(1)}$(100) |
|---|---|---|
| 100.000 000 20 | 100.000 000 20 | 100.000 000 00 |
| 101.000 000 16 | 101.000 000 16 | 101.000 000 00 |
| 102.000 000 20 | 102.000 000 20 | 102.000 000 00 |
| 103.000 000 18 | 103.000 000 18 | 103.000 000 00 |

$q = 3$

| $x_i$(51) | $x_i^{(0)}$(100) | $x_i^{(1)}$(100) |
|---|---|---|
| 1 000.000 000 1 | 1 000.000 000 2 | 0 999.999 981 8 |
| 1 000.999 999 8 | 1 000.999 999 8 | 1 000.999 981 8 |
| 1 002.000 000 2 | 1 002.000 000 2 | 1 001.999 981 7 |
| 1 003.000 000 0 | 1 003.000 000 0 | 1 002.999 981 8 |

$q = 4$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ | $x_i^{(2)}(100)$ |
|---|---|---|---|
| 10 000.001 820 | 10 000.001 821 | 10 000.000 000 | 10 000.000 000 |
| 10 001.001 817 | 10 001.001 817 | 10 001.000 000 | 10 001.000 000 |
| 10 002.001 821 | 10 002.001 821 | 10 001.999 999 | 10 002.000 000 |
| 10 003.001 819 | 10 003.001 819 | 10 002.999 999 | 10 003.000 000 |

$q = 5$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ |
|---|---|---|
| 100 000.181 91 | 100 000.181 92 | 100 000.000 00 |
| 100 001.181 88 | 100 001.181 88 | 100 001.000 00 |
| 100 002.181 92 | 100 002.181 92 | 100 002.000 00 |
| 100 003.181 90 | 100 003.181 90 | 100 003.000 00 |

$q = 6$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ | $x_i^{(2)}(100)$ |
|---|---|---|---|
| 1 000 018.190 2 | 1 000 018.190 3 | 0 999 999.999 7 | 1 000 000.000 0 |
| 1 000 019.189 9 | 1 000 019.189 9 | 1 000 000.999 7 | 1 000 001.000 0 |
| 1 000 020.190 3 | 1 000 020.190 3 | 1 000 001.999 7 | 1 000 002.000 0 |
| 1 000 021.190 1 | 1 000 021.190 1 | 1 000 002.999 7 | 1 000 003.000 0 |

$q = 7$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ | $x_i^{(2)}(100)$ |
|---|---|---|---|
| 10 001 819.176 | 10 001 819.176 | 09 999 999.669 | 10 000 000.000 |
| 10 001 820.172 | 10 001 820.173 | 10 000 000.669 | 10 000 001.000 |
| 10 001 821.176 | 10 001 821.176 | 10 000 001.669 | 10 000 002.000 |
| 10 001 822.174 | 10 001 822.175 | 10 000 002.669 | 10 000 003.000 |

$q = 8$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ | $x_i^{(2)}(100)$ |
|---|---|---|---|
| 100 182 149.38 | 100 182 149.38 | 099 999 668.22 | 100 000 000.60 |
| 100 182 150.35 | 100 182 150.35 | 099 999 669.22 | 100 000 001.61 |
| 100 182 151.39 | 100 182 151.38 | 099 999 670.22 | 100 000 002.61 |
| 100 182 152.36 | 100 182 152.37 | 099 999 671.22 | 100 000 003.61 |

$q = 9$

| $x_i(51)$ | $x_i^{(0)}(100)$ | $x_i^{(1)}(100)$ | $x_i^{(2)}(100)$ |
|---|---|---|---|
| 1 000 000 000.3 | 1 037 735 849.4 | 0 999 301 188.0 | 100 000 000.00 |
| 1 000 000 001.0 | 1 037 735 850.1 | 0 999 301 189.0 | 100 000 001.00 |
| 1 000 000 001.8 | 1 037 735 851.9 | 0 999 301 190.0 | 100 000 002.00 |
| 1 000 000 003.0 | 1 037 735 852.1 | 0 999 301 191.0 | 100 000 003.00 |

| $x_1^{(3)}(100)$ | $x_2^{(3)}(100)$ |
|---|---|
| 1 000 026 370.3 | 0 999 999 511.6 |
| 1 000 026 371.2 | 0 999 999 512.6 |
| 1 000 026 372.3 | 0 999 999 513.6 |
| 1 000 026 373.2 | 0 999 999 514.6 |

| $x_1^{(6)}(100)$ | $x_2^{(6)}(100)$ |
|---|---|
| 1 000 000 018.4 | 0 999 999 999.3 |
| 1 000 000 019.4 | 1 000 000 000.3 |
| 1 000 000 020.4 | 1 000 000 001.3 |
| 1 000 000 021.4 | 1 000 000 002.3 |

| $x_1^{(9)}(100)$ |
|---|
| 1 000 000 000.0 |
| 1 000 000 001.0 |
| 1 000 000 002.0 |
| 1 000 000 003.0 |

Several remarks about these results are in order. It is not to be expected that the zeroth order solutions of Routine 100 should agree with the solutions of Routine 51 since the conversion routine which reads the matrix into ILLIAC is slightly different in the two programs. The one in Routine 51 is accurate to $\pm 2^{-39}$, that in Routine 100 to $\pm 2^{-40}$. It is to be noted that as the equations become more poorly conditioned a greater number of iterations are necessary before Routine 100 converges to the correct answer.

The deterioration of the Routine 100 solution for $q = 3$ upon iteration and the already excellent Routine 51 solution for $q = 9$ demand especial attention. Both of these effects are traceable to the rounding off of the scaling factors to 39 binary digits in the machine. For the case $q$, a scaling factor $10^{-q-1}$ is used by both Routine 51 and Routine 100 for the first iteration in accordance with equation (5). In addition Routine 100 uses the scaling factor again on each subsequent iteration to form the residues and to reduce the resultant corrections to the uniform scale of the zeroth order solutions. The machine representation of the scaling factor was examined for each of the above cases. For $q = 3$, the scaling factor $10^{-4}$ must be rounded down by six parts in the 11th sexadecimal digit when the number is rounded to ILLIAC's register size of 10 sexadecimal digits. This is the largest rounding-off error encountered in the sequence. This accounts for the deterioration of Routine 100's first iterate since the correction is multiplied by this scaling factor before being added to the zeroth order solution. For $q = 9$, the scaling factor $10^{-10}$ when represented to 10 sexadecimal digits is also exact to 11 sexadecimal digits, thus accounting for the already excellent Routine 51 solution. Such effects would argue for the use of binary scaling factors during the carrying out of the back substitution. However, if the answers were presented with such factors they would not be immediately useful; if the answers were reconverted with the appropriate decimal scaling factors before presentation not only would additional memory space be required for such reconversion but the imprecision of these decimal scale factors would be introduced.

Recent changes in the ILLIAC have resulted in an increased efficiency in the order alteration portions of a program. Various routines of the Program Library are being rewritten to utilize this increased efficiency. When Routine 100 is rewritten, sufficient memory space will be gained so that extra orders can be added to the program which will allow the residues to be computed more accu-

rately. This should speed the convergence and should eliminate most of the effect of imprecision in the scaling factor.

JAMES N. SNYDER

University of Illinois
Urbana, Illinois

1. D. J. WHEELER & J. P. NASH, "Digital and Analogue Computers and Computing Methods." Symposium at the 18th Applied Mechanics Division Conference of the American Society of Mechanical Engineers, University of Minnesota, June 18–20, 1953.
2. D. J. WHEELER, *The Automatic Linear Equation Solver*, University of Illinois Computer Library Routine No. 51.
3. J. N. SNYDER, *The Complete Linear Equation Solver*, University of Illinois Computer Library Routine No. 100.

# The Use of Iterative Methods for finding the Latent Roots and Vectors of Matrices

In a recent note in *MTAC* E. BODEWIG [1] presented what he claimed was "a practical refutation of the iteration method for the algebraic eigenproblem." In my opinion this note gave an entirely misleading impression of the value of the iterative method. Moreover an example was chosen as the basis of this refutation which so far from serving the purpose for which it was used, is in fact quite well suited to the iterative method provided it is used in a flexible manner. The iterative method, supplemented by a number of simple devices for accelerating convergence, has been used very effectively on the Pilot ACE to find the latent roots and vectors of a very large number of matrices, symmetric and unsymmetric, real and complex, up to orders as high as 60. Very high accuracy has been achieved even when many or all of the latent vectors of a matrix have been wanted. The details of the method used have been described in a recent paper [2] by WILKINSON, but since that paper was written a magnetic drum store has been added to the Pilot ACE and the speed of iteration has thereby been considerably increased particularly for the larger matrices. The addition of the drum has also led to modifications in the details of the programme which have made it much more satisfactory to use. For this reason a general description of the programme is given below. The note includes an assessment of the value of iteration and concludes with the results achieved with it on Bodewig's example.

An understanding of the iterative programme will be aided by a description of the one or two facilities provided on the Pilot ACE, which are employed therein. There is a register, called the input register, which stores one of the standard words of 32 binary digits, into which a number may be inserted manually by means of 32 keys. The number thus inserted is displayed on a set of 32 lights and this number may be changed at any time during computation by the operator. The input register is addressable in the same way as all the other storage registers and the machine has access in 32 microseconds to the number held there, but it cannot send a word *to* the input register. There is a second register, the output register, the contents of which are also displayed on a set of 32 lights. The machine may send a number to the output register in 32 microseconds, but it cannot read the number stored in it. The machine is also equipped with a monitoring device on which are displayed the contents of 32 consecutive storage registers. The

monitor may be made to display different groups of 32 words by the appropriate setting of a number of keys. The 32 words displayed on the monitor appear on 32 rows in the form shown in fig. 1 where the first three lines only are shown with words of 10 binary digits instead of 32 for convenience.

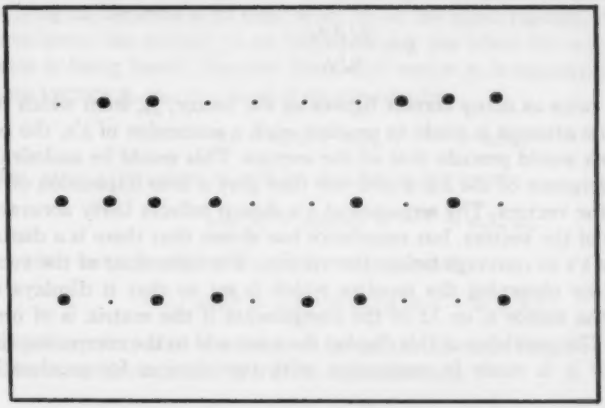



FIG. 1.

The matrix $A$ of which the latent roots are required is stored on the drum, and the present drum can deal with matrices up to order 60. Information may be transferred from the drum to the high speed store in blocks of 32 words at a time, each such transfer taking place in a time which, in this programme, is only 10% of the time taken to use that block of information. The machine starts with an arbitrary vector $y_0$ (the programme is arranged to read in a good initial guess if one is known) and from it forms two sequences of vectors $y_s$ and $z_s$ defined by the relations

$$z_s = (A - pI)y_s$$

$$y_{s+1} = z_s/(\text{element of } z_s \text{ of maximum modulus})$$

where $p$ is the number set up on the input register. The normalisation of $z_s$ at each stage is carried out in order to keep numbers within range and it has the effect that the largest element in each $y_s$ is unity. Since $p$ is read in at the beginning of each iteration only, it may be changed at any time during an iteration without affecting the value which is used for the remainder of that iteration. The vectors $y_s$ will tend to the latent vector, $x$, corresponding to that value of $\lambda$ (assumed real) which is either the algebraically largest latent root or the algebraically smallest, according to the value of $p$. The value of $p$ may also be chosen to improve the rate of convergence. Although $p$ is set up as a binary number this is not particularly inconvenient because there is no need to choose $p$ with any great precision, so that most of the less significant digits are left as zero. The progress of the convergence may be studied by observing the successive approximations to $\lambda$ which appear on the output register. Since the time taken to send a number to the output register is 32 microseconds this display does not add

appreciably to the time of an iteration, as would printing successive values of $\lambda$. As $y_s$ tends to the latent vector, the largest element of $z_s$ tends to $(\lambda - p)$ and the approximation to $\lambda$ which is sent to the output register at the end of each iteration is found by adding $p$ to this largest element. It is well known that for a symmetric matrix the RAYLEIGH value given by

$$\lambda = \frac{y_s' A y_s}{y_s' y_s}$$

has, in general, twice as many correct figures as the vector, $y_s$, from which it is calculated, but no attempt is made to produce such a succession of $\lambda$'s, the convergence of which would precede that of the vectors. This would be undesirable in that the convergence of the $\lambda$'s would not then give a true impression of the convergence of the vectors. The sequence of $\lambda$'s chosen reflects fairly accurately the convergence of the vectors, but experience has shown that there is a distinct tendency for the $\lambda$'s to converge before the vectors. The behaviour of the vector may be studied by observing the monitor which is set so that it displays the components of the vector $z_s$ or 32 of the components if the matrix is of order greater than 32. The provision of this display does not add to the computing time and great use of it is made in connection with two devices for accelerating convergence.

At the beginning of the iteration process, all the digits of all the components of $z_s$ will change with each iteration, but after a few iterations the more significant digits of $z_s$ will become constant and this is easily observed by the operator. Since the values of $p$ set up on the input register will always contain zeros in the lower digital positions, these positions are effectively free and may be used as further controls on the programme. The two least significant digits of the input register, which are called P1 and P2, are in fact used in the following manner. If at any time the P1 digit is set up on the input register then at the end of the current iteration, the machine takes the last three iterated vectors

$$y_{s-2}, \quad y_{s-1} \quad \text{and} \quad y_s$$

and from them forms the vector $Y$, the $i^{\text{th}}$ component, $Y^i$, of which is given in terms of the $i^{\text{th}}$ components of the three $y$ vectors by the relation

$$Y^i = \frac{y^i_{s-2} y^i_s - (y^i_{s-1})^2}{y^i_{s-2} - 2y^i_{s-1} + y^i_s}.$$

This vector $Y$ is referred to as the AITKEN vector derived from $y_{s-2}$, $y_{s-1}$ and $y_s$. In general, as described by WILKINSON [2], the vector $Y$ will be a much improved approximation to the latent vector. In order to be able to form the Aitken vector at any time, it is only necessary to store one extra vector. Suppose the successive vectors $y_s$ are stored in the set of registers A and the vectors $z_s$ in the set of registers B. Then on completing an iteration, if an application of the Aitken process is not required, $y_s$ is transferred from A to a set of registers C and $z_s$ is normalised to $y_{s+1}$ which is then stored in A. If at the end of the next iteration an application of the Aitken process has been requested by the operator then the transfer from A to C does not take place and when $z_{s+1}$, which is in B, is normalised

to $y_{s+2}$ this is written over $z_{s+1}$ instead of in A. Thus we have $y_s$ in C, $y_{s+1}$ in A and $y_{s+2}$ in B and this is sufficient for an application of the Aitken process. The Aitken process may be used as often or as little as the operator demands, though there must be at least three iterations between each application.

The P2 digit is used to modify the operation of the programme as follows. If during an iteration a P2 digit is set up on the input register, then the machine stores away the current $y_s$ on the drum for use when the subdominant latent vector is being found. Suppose the initial vector $y_0$ is expressed in terms of the latent vectors $x_1, x_2, \cdots, x_n$ of $A$ by the relation

$$y_0 = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Then, after $s$ iterations, $y_s$ will be parallel to the vector

$$\alpha_1 \lambda_1^s x_1 + \alpha_2 \lambda_2^s x_2 + \cdots + \alpha_n \lambda_n^s x_n.$$

and after some iterations we will have in general

$$\alpha_1 \lambda_1^s \gg \alpha_2 \lambda_2^s \gg \alpha_3 \lambda_3^s \gg \cdots \gg \alpha_n \lambda_n^s.$$

This means that after a number of iterations the main impurity in $y_s$ will consist of the $x_2$ component and $y_s$ will be approximately

$$x_1 + \epsilon x_2.$$

When the iteration for the dominant vector $x_1$ is complete a matrix of order one less is formed by a process of root removal and it is shown by WILKINSON [2] that

$$(y_s - x_1)$$

where $y_s$ is the vector which was stored on the drum, will, in general, be a very good initial approximation to the latent vector of the reduced matrix. The P2 digit will be used once only during the process of iterating for one vector, and usually the time that is chosen is when about half of the binary digits in the successive $y_s$ are constant. Iteration for each latent vector is continued until all the digits of $y_s$ are constant or the last few digits are going through a repeated cycle of values. This is most important because if we are to use successive root removals, the latent vectors must be as accurate as possible. The decision when iteration for a latent vector is complete is made by the operator and the programme is designed to accept a signal advising it when this is so. The process of reducing the matrix [2] then takes place and the machine begins to iterate with the reduced matrix for the next vector, determining its initial guess as described above.

The last two techniques described are invaluable and are sometimes so successful that, starting from the good initial guess, 3 iterations plus one application of the Aitken process are sufficient for complete convergence.

One or two comments on the role played by the operator might be made at this point. In the first place it should be stressed that at no time does the success of the programme depend critically on the operator's performance. A failure to produce the best value of $p$, to apply the Aitken process or to store away a vector at the optimum time merely means that one does not gain as much as one might

have done. It is undoubtedly true that an operator becomes rather more proficient with the programme after a little experience with using it but the operators employed on the Pilot ACE are in no sense of the word skilled mathematicians, and I think it is generally agreed that the programme is much more entertaining than any other used on ACE.

In my opinion the "algebraic eigenproblem" is not a problem for which there is a universal solution. There are, in practice, many different requirements. The matrix may be symmetric or unsymmetric and in the latter case, the roots, real or complex. All the roots may be wanted, several dominant roots, the few smallest roots or the roots, if any, in a given range. Again we may or may not want the vectors. Since all machines have stores of finite size often divided up into high speed and auxiliary sections, storage considerations often have a vitally important part to play.

All possible combinations of the above requirements have been met in using the Pilot ACE and the iterative process has proved to be about the most useful weapon at our disposal but it is certainly not the only method we use. For example if all the roots, but not the vectors, of a symmetric matrix are wanted the JACOBI process [3] is undoubtedly better than iteration unless the separation of roots is quite exceptionally good. If a few isolated roots of a symmetric matrix, but not the vectors, are wanted, then the method described by W. GIVENS [4] is almost certainly the most effective of known methods. The iteration method has been used mainly on unsymmetric matrices and most commonly where several of the dominant roots and the vectors have been wanted but it would be used equally on symmetric matrices where the requirement was the same.

As regards the separation of successive roots, Bodewig's statement that a separation of four to one is necessary is very wide of the mark. For some of the larger matrices, of orders 50 to 60, which we have received from aircraft firms, separations of one part in ten have been quite common and much worse separations have been dealt with satisfactorily using the above accelerating techniques. Among smaller matrices quite bad separations have been treated very successfully. Below are given the roots of three matrices A, B and C for which separation is bad. These three examples are chosen from a group of 30 unsymmetric matrices of orders 10 to 15 all of which gave separations comparable with those in the examples.

| Roots of Matrix A | Matrix B | Matrix C |
|---|---|---|
| 0.2144582 | 0.2155673 | 0.1562830 |
| 0.4644432 | 0.4826537 | 0.3571287 |
| 0.6813217 | 0.7523862 | 0.9520832 |
| 0.9824573 | 1.0217583 | 1.2017653 |
| 1.2457623 | 1.5162837 | 1.8635762 |
| 1.7428765 | 2.0256132 | 1.8801257 |
| 2.1281317 | 2.6578326 | 2.4613258 |
| 2.6315722 | 4.1243587 | 2.9980751 |
| 3.2458126 | 4.5625830 | 3.5287125 |
| 5.4916793 | 5.5612571 | 4.0165324 |
| 5.5491920 | 5.6013757 | 4.5238175 |
| 5.5695580 | | |

In matrix A, for example, the three dominant roots are separated by one part in 270 and one part in 100 respectively. It was immediately obvious that convergence for the dominant root was very slow so that an attempt was made to find the roots in the reverse order by an appropriate choice of $p$. A value of $p$ was chosen which was appreciably greater than half the value of the initial approximations which had been sent to the output register when iterating with $p = 0$. The values of $p$ which were used were not kept but it is probable that the root 0.2144582, which was found first, was found by iterating with $(A - pI)$ where $p$ was between 3 and 3.5. The separation between the two dominant roots of $(A - 3I)$, for example, is about one part in ten and this with the accelerating process is quite adequate. The three largest roots were found last and by this time the reduced matrix was of order 3, so that iterations were taking place at the rate of 30 per second. Choosing $p$ to be near the approximations to $\lambda$ which appeared in the output register also gave quite good separation so that the largest three roots were found very quickly. For matrix C the two roots 1.8635762 and 1.8801257 were found last.

It is popularly believed that the eigenvectors obtained after several root removals tend to be fairly inaccurate. In this connection our experience on the Pilot ACE with several hundred matrices (literally several thousand if matrices of order less than 10 are included!) on which the iteration method has been used is most interesting. If we define the residuals corresponding to a latent root $\lambda$ and a vector $x$ as the components of

$$Ax - \lambda x$$

then in no example has any residual been greater than $20 \times 2^{-29}$ (i.e., approximately $37 \times 10^{-9}$) where it is assumed that the elements of $A$ are of order unity. Moreover more than 90% of all residuals are in fact below $5 \times 2^{-29}$ and a set of latent vectors with all its residuals consisting of either zeros, $\pm 2^{-29}$, $\pm 2^{-28}$ is far from uncommon. The main reasons for the accuracy are that iteration for each vector is taken to the limit and scalar products are accumulated with all the digits produced by each multiplication (N.B., the elements of the matrix A and the vectors $y_s$ are single length numbers). No other method which we have used has produced such consistently accurate vectors. Processes in which the latent roots are found from the characteristic equation have, in our experience proved to be deceptively satisfactory when dealing with small matrices. For larger order matrices it very easily happens that such precautions are necessary in forming the characteristic equation that what appears to have been a very fast and effective method for small matrices, proves unexpectedly tiresome for those of higher order. Determining the latent vectors accurately, particularly in the case of unsymmetric matrices is even more hazardous, and it is therefore very misleading to try to assess the relative efficiency of methods by counting the number of multiplications they require. Bodewig remarks that it is possible to find the latent roots and vectors to any accuracy from the characteristic equation but it should be remembered that the accuracy is in fact limited by that of the characteristic equation which has been calculated and that when roots are close

together a given accuracy in the characteristic equation does not produce the same accuracy in the roots.

It is interesting to see how the Pilot ACE programme for iteration dealt with the example given by Bodewig. The coefficients of the matrix were stored in the machine to a precision of 29 binary figures (or 8.7 decimals) for a reason which is associated with the details of the programme. Iteration was started with $p = 0$ and it was immediately obvious that convergence, which one would expect to be almost instantaneous for a matrix of order 4, was in fact proceeding comparatively slowly. As a trial value, to speed convergence, $p = 2$ was set up on the input register. Convergence then took place almost immediately so that no thought was given to finding a better value of $p$. All four roots were found in a few seconds iteration time and the roots and vectors, after normalising to agree with Bodewig's method, were as given below.

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix}$$

$\lambda_1 = -8.02857835$ $\lambda_2 = +7.93290471$ $\lambda_3 = +5.66886437$ $\lambda_4 = -1.57319073$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|
| +1.00000000 | +1.00000000 | +1.00000000 | +1.00000000 |
| +2.50146029 | +0.37781815 | +0.95700150 | −0.90709211 |
| −0.75773064 | +1.38662122 | −1.42046822 | −0.37759122 |
| −2.56421169 | +0.34880573 | +1.74331690 | −0.38333124 |

The vectors agree, apart from the end figure of one or two components, with those given by Bodewig, except for the final component of $x_4$ where the two results differ by 0.00001886. It is easy to satisfy oneself that Bodewig's result is in error, and it is probable that his results were copied incorrectly. The only peculiarity displayed by Bodewig's example was that its two dominant roots were almost equal and opposite in value and this is not a difficult situation. Indeed even if the roots had been exactly equal and opposite in value there would have been no difficulty whatever. If the two dominant roots had had the values in the example but with the same sign the situation would have been just a little less satisfactory. Then the root −1.573 etc., would have been found first and the root 5.669 second. The matrix obtained after two reductions would have been of order two, so that iteration would have been extremely fast and a choice of $p$ anywhere near 8.0 would have given almost immediate convergence. Such a choice of $p$ would have been suggested by the approximations to $\lambda$ obtained on the output register. Besides being a poor choice on which to base a refutation of the iteration process, Bodewig's example has other weaknesses which make it a rather poor one for estimating the relative efficiency of methods. In the first place it is of very low order so that it does not reveal the difficulties of preserving accuracy which arise with larger matrices and secondly, it has small integer coefficients. This

means, for example, that if we use a method in which we find the characteristic equation we will almost certainly obtain the latter exactly, which will not be true if the coefficients were numbers with several digits.

J. H. WILKINSON

National Physical Laboratory
Teddington, Middlesex
England

This note is published with the permission of the Director of the National Physical Laboratory.

1. E. BODEWIG, "A practical refutation of the iteration method for the algebraic eigenproblem," *MTAC*, v. 8, 1954, p. 237–239.
2. J. H. WILKINSON, "The calculation of the latent roots and vectors of matrices on the Pilot Model of the ACE," Cambridge Phil. Soc., *Proc.*, v. 50, 1954, p. 536–566.
3. C. G. J. JACOBI, "Über ein leichtes Verfahren, die in der Theorie der Säkularstörungen vorkommenden Gleichungen numerisch aufzulösen," *J. reine angew. Math.* 30, 1846, p. 51–94.
4. W. GIVENS, "Numerical computation of the characteristic values of a real symmetric matrix," Oak Ridge National Laboratory. ORNL1574.

## TECHNICAL NOTES AND SHORT PAPERS

### The Specific Heat Function for a Two-Dimensional Continuum

**Numerical Values of**

$$\frac{C_2}{C_\infty} = \frac{6}{x^2}\int_0^x \frac{\xi^3 d\xi}{e^\xi - 1} - \frac{2x}{e^x - 1}.$$

This function which appears in the theory of low-temperature specific heats of two-dimensional (layer) structures [1] was computed as follows:

(1) For $0 \leqslant x \leqslant 2.0$, the formula

$$\frac{C_2}{C_\infty} = 1 - \frac{x^2}{24} + \frac{x^4}{720} - \frac{x^6}{24{,}192} + \frac{x^8}{864{,}000} - \frac{x^{10}}{31{,}933{,}440} + \cdots$$

was used. The maximum error (using seven terms) is no greater than $0.5 \times 10^{-5}$.

(2) For $2.0 \leqslant x \leqslant 16.0$, the formula

$$\frac{C_2}{C_\infty} = \frac{14.424684}{x^2} - 6x\sum_{n=1}^{\infty} e^{-nx}\left(\frac{1}{nx} + \frac{2}{(nx)^2} + \frac{2}{(nx)^3}\right) - \frac{2x}{e^x - 1}$$

was used. The maximum error was approximately $2 \times 10^{-6}$.

The value $C_\infty = 3R = 5.9616$ cal mol$^{-1}$ deg$^{-1}$ was used throughout [2].

| $x$ | $C_2/C_\infty$ | $C_2$ |
|---|---|---|
| 0.0 | 1.00000 | 5.9616 |
| 0.1 | 0.99958 | 5.95911 |
| 0.2 | .99833 | 5.95167 |
| 0.3 | .99625 | 5.93925 |
| 0.4 | .99333 | 5.92186 |
| 0.5 | .98959 | 5.89955 |

| $x$ | $C_2/C_\infty$ | $C_2$ |
|---|---|---|
| 0.6 | .98502 | 5.87227 |
| 0.7 | .97961 | 5.84006 |
| 0.8 | .97340 | 5.80300 |
| 0.9 | .96632 | 5.76081 |
| 1.0 | .95843 | 5.71378 |
| 1.1 | .95155 | 5.67274 |
| 1.2 | .94276 | 5.62037 |
| 1.3 | .93336 | 5.56432 |
| 1.4 | .92337 | 5.50478 |
| 1.5 | .91284 | 5.44198 |
| 1.6 | .90179 | 5.37610 |
| 1.7 | .89026 | 5.30737 |
| 1.8 | .87829 | 5.23601 |
| 1.9 | .86592 | 5.16224 |
| 2.0 | .85318 | 5.08629 |
| 2.1 | .84024 | 5.00919 |
| 2.2 | .82677 | 4.92890 |
| 2.3 | .81327 | 4.84841 |
| 2.4 | .79936 | 4.76548 |
| 2.5 | .78526 | 4.68138 |
| 2.6 | .77111 | 4.59707 |
| 2.7 | .75689 | 4.51225 |
| 2.8 | .74243 | 4.42607 |
| 2.9 | .72802 | 4.34016 |
| 3.0 | .71353 | 4.25379 |
| 3.1 | .69901 | 4.16719 |
| 3.2 | .68454 | 4.08097 |
| 3.3 | .67014 | 3.99511 |
| 3.4 | .65582 | 3.90974 |
| 3.5 | .64159 | 3.82488 |
| 3.6 | .62746 | 3.74068 |
| 3.7 | .61347 | 3.65725 |
| 3.8 | .59961 | 3.57461 |
| 3.9 | .58594 | 3.49313 |
| 4.0 | .57240 | 3.41244 |
| 4.1 | .55910 | 3.33315 |
| 4.2 | .54597 | 3.25483 |
| 4.3 | .53306 | 3.17787 |
| 4.4 | .52037 | 3.10225 |
| 4.5 | .50791 | 3.02793 |
| 4.6 | .49568 | 2.95502 |
| 4.7 | .48368 | 2.88352 |
| 4.8 | .47193 | 2.81343 |
| 4.9 | .46041 | 2.74480 |
| 5.0 | .44915 | 2.67766 |

| $x$ | $C_2/C_\infty$ | $C_2$ |
|---|---|---|
| 5.1 | .43814 | 2.61202 |
| 5.2 | .42738 | 2.54789 |
| 5.3 | .41687 | 2.48521 |
| 5.4 | .40661 | 2.42403 |
| 5.5 | .39659 | 2.36434 |
| 5.6 | .38683 | 2.30611 |
| 5.7 | .37731 | 2.24937 |
| 5.8 | .36803 | 2.19404 |
| 5.9 | .35900 | 2.14019 |
| 6.0 | .35019 | 2.08771 |
| 6.1 | .34163 | 2.03664 |
| 6.2 | .33328 | 1.98691 |
| 6.3 | .32517 | 1.93853 |
| 6.4 | .31728 | 1.89152 |
| 6.5 | .30961 | 1.84577 |
| 6.6 | .30215 | 1.80129 |
| 6.7 | .29490 | 1.75805 |
| 6.8 | .28785 | 1.71604 |
| 6.9 | .28100 | 1.67519 |
| 7.0 | .27434 | 1.63554 |
| 7.1 | .26788 | 1.59699 |
| 7.2 | .26160 | 1.55954 |
| 7.3 | .25550 | 1.52318 |
| 7.4 | .24959 | 1.48797 |
| 7.5 | .24382 | 1.45353 |
| 7.6 | .23823 | 1.42021 |
| 7.7 | .23280 | 1.38786 |
| 7.8 | .22753 | 1.35643 |
| 7.9 | .22241 | 1.32591 |
| 8.0 | .21744 | 1.29628 |
| 8.1 | .21258 | 1.26733 |
| 8.2 | .20792 | 1.23954 |
| 8.3 | .20337 | 1.21239 |
| 8.4 | .19894 | 1.18602 |
| 8.5 | .19465 | 1.16041 |
| 8.6 | .19048 | 1.13554 |
| 8.7 | .18642 | 1.11137 |
| 8.8 | .18248 | 1.08789 |
| 8.9 | .17866 | 1.06508 |
| 9.0 | .17494 | 1.04291 |
| 9.1 | .17132 | 1.02136 |
| 9.2 | .16781 | 1.00043 |
| 9.3 | .16440 | .98009 |
| 9.4 | .16108 | .96030 |

| $x$ | $C_2/C_\infty$ | $C_2$ |
|---|---|---|
| 10.0 | .14302 | .85262 |
| 10.5 | .13006 | .77535 |
| 11.0 | .11873 | .70779 |
| 11.5 | .10897 | .64965 |
| 12.0 | .09998 | .59604 |
| 12.5 | .09220 | .54965 |
| 13.0 | .08528 | .50840 |
| 13.5 | .07910 | .47157 |
| 14.0 | .07357 | .43857 |
| 14.5 | .06859 | .40890 |
| 15.0 | .06411 | .38218 |
| 15.5 | .06003 | .35789 |
| 16.0 | .05634 | .33589 |

M. DANK
S. W. BARBER

Owens-Illinois Glass Co.
Toledo, Ohio

1. V. V. TARASOV, "Teoriia Teploemkosti Tsepnykh I Sloistykh Struktur" [The Theory of Heat Capacity of Chain and Linear Structures], *Zhurnal Fiz. Khimii*, XXIV, 1950, p. 111–128.
2. J. W. M. DUMOND & E. RICHARD COHEN, "Least-squares adjusted values of the atomic constants as of December, 1950," *Phys. Rev.*, v. 82, p. 555–556.

## A Continued Fraction for $e^x$

Let $A_n/B_n$ denote the convergents to the continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots.$$

It is well known [1] that the continued fraction

$$b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2} - \frac{a_2 a_3 b_4}{(b_2 b_3 + a_3) b_4 + b_2 a_4} - \frac{a_4 a_5 b_2 b_6}{(b_4 b_5 + a_5) b_6 + b_4 a_6} - \frac{a_6 a_7 b_4 b_8}{(b_6 b_7 + a_7) b_8 + b_6 a_8} - \cdots$$

has convergents $A_{2n}/B_{2n}$.

From the continued fraction of Gauss, one can obtain [2]

$$e^x = 1 + \frac{x}{1} - \frac{x/1\cdot 2}{1} + \frac{x/2\cdot 3}{1} - \frac{x/2\cdot 5}{1} + \frac{x/2\cdot 5}{1} - \cdots.$$

For this expansion, we have $b_0 = 1$, $a_1 = x$, $a_{2n} = -x[2(2n-1)]$, $a_{2n+1} = x/[2(2n+1)]$, and $b_n = 1$ $(n = 1, 2, 3, \cdots)$. Thus, we obtain

$$-a_{2n}a_{2n+1}b_{2n-2}b_{2n+2} = x^2/[4(4n^2-1)],$$

and

$$(b_{2n}b_{2n+1} + a_{2n+1})b_{2n+2} + b_{2n}a_{2n+2} = 1, \quad (n = 2, 3, \cdots);$$

and so

$$e^x = 1 + \frac{x}{1 - x/2} + \frac{x^2/(4\cdot 3)}{1} + \frac{x^2/(4\cdot 15)}{1} + \cdots + \frac{x^2/\{4[4(\nu-1)^2 - 1]\}}{1} + \cdots.$$

A convenient equivalent form is

$$e^x = 1 + \frac{x}{1 - x/2} + \frac{x^2/4}{3} + \frac{x^2/4}{5} + \cdots + \frac{x^2/4}{2\nu - 1} + \cdots.$$

These expansions converge quite rapidly for all $x$. For example, if $-1 \leqslant x \leqslant 1$, we have $e^x - A_4/B_4 < .000084$, $e^x - A_5/B_5 < .000000033$, and $e^x - A_6/B_6 < .000000000081$.

N. MACON [3]

Alabama Polytechnic Institute
Auburn, Alabama

This note was written while the author was a summer participant at the Oak Ridge National Laboratory, 1954.

1. See, for example, OSKAR PERRON, *Die Lehre von den Kettenbrüchen*, 2nd ed., Teubner, Leipzig, 1929, p. 201.
2. PERRON, *op. cit.*, p. 312. Mr. Garwick also notes that this is credited to Thiele, see, for example, N. E. Nørlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924, Chapter 15, p. 415–455, especially p. 454.
3. The referee has noted that this work has been done independently by J. V. GARWICK of the Norwegian Defense Research Establishment.

## REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

**78[C, D, E].**—FRIEDRICH LÖSCH, *Siebenstellige Tafeln der elementaren transzendenten Funktionen*. Springer, Berlin-Göttingen-Heidelberg, 1954, viii + 335 p. 27 cm. DM 49.80.

This volume is meant to supersede the 1926 table of K. HAYASHI [1], now out of print. The elementary transcendental functions are $\sin x$, $\cos x$, $\tan x$, $\sinh x$, $\cosh x$, $\tanh x$, $\ln x$, $e^x$, $e^{-x}$, $\arcsin x$, $\arctan x$, $\operatorname{arcsinh} x$, $\operatorname{arctanh} x$, together with $\varphi$, the angle corresponding to the arc $x$. In Table I (40 pages) these are given to 9D for $x = 0(.0001)0.1$, together with first differences (except for $\ln x$, and, of course, $\varphi$). In Table II (277 pages) these are given to 7D for $x = 0.1(.0005)3.15$, with (*double*) first differences for all functions (but $\varphi$) and for $x = 3(.01)10(.1)20$, without differences. For $x > 1$, $\operatorname{arccoth} x$ is tabulated in place of $\operatorname{arctanh} x$ and $\operatorname{arccosh} x$ in place of $\arcsin x$. There are supplementary tables of $\tan x$ to 7D for $x = 1.5680(.0001)1.5730$, $\operatorname{arctanh} x$ for $x = 0.9980(.0001)1$, $\operatorname{arccoth} x$ for $x = 1(.0001)1.0020$.

In Table III, there are given $\sin x$, $\cos x$, $\ln x$, $\operatorname{arcsinh} x$, $\operatorname{arccosh} x$ to 7D, $e^{\pm x}$ to 7S, for $x = 0(1)100$. Table IV gives $\frac{1}{2}n\pi$ to 12D for $n = 0(1)100$. Table V gives $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$ to 7D for $x = 0(.001)0.5$. Table VI gives $\exp(\pm\frac{1}{2}\pi x)$, $\sinh \frac{1}{2}\pi x$, $\cosh \frac{1}{2}\pi x$ to 7D for $x = 0(.01)2$. Table VII gives $\exp(\pm\pi x/180)$, $\sinh(\pi x/180)$, $\cosh(\pi x/180)$ to 7D for $x = 0(1)180$. Table VIII gives $\varphi$ in degrees

(12D) and in degrees, minutes and seconds (8D) for $x = 0(.0001).001(.001)$ $.01(.01).1(.1)1(1)10(10)100$. Table IX gives $x$ in radians to 12D for $\varphi = 0(1°)90°$, for $\varphi = 0(1')1°$ and for $\varphi = 0(1'')1'$. Table X gives a collection of constants to 15D, the exact values of the first twenty Bernoulli numbers and the first 15 factorials; also $(n!)^{-1}$ to 15S, $\log_{10} n!$ to 15D and the binomial coefficients $\binom{n}{r}$ for $n = 1(1)15, r = 1(1)[\frac{1}{2}n]$.

The volume is excellently printed on rather large pages; it is, however, expensive. The Introduction gives worked examples, which cover typical cases. In general the error in linear interpolation (in Table I, II) is at most two units; in cases where it is larger, attention is called by use of italics for differences or by their omission.

It is stated that considerable parts of the tables were computed especially for this volume; for the remainder use was made of available tables and interpolation in such tables; the calculations were made to 9 or 10D and checked by differencing before rounding to 7 or 9D. The new material is mainly that concerned with the inverse hyperbolic functions and the inverse tangent; some of the hyperbolic material is available in the Harvard volume [2].

Two pages of Table I and two of Table II were checked by comparison with recognized sources and new calculations (made on SEAC): no discrepancies were noted.

Users will have to be wary of noting the leading figures of values and differences at the head (and foot) of the columns and the changes which occur in them.

J. T.

1. K. HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbel-funktionen und deren Produkte sowie der Gammafunktion*. Springer, Berlin, 1926.
2. HARVARD UNIVERSITY, Computation Laboratory, Annals, vol. 20, *Tables of inverse hyperbolic functions*, 1949.

**79[D, I].**—OWEN R. MOCK, *A table of sines and cosines of* $\frac{\pi}{2} \cdot \frac{k}{256}$. 13 typewritten leaves deposited in UMT FILE.

These tables, constructed for Fourier analysis and synthesis with data from binary sources list sin and cos to 25D for $k = 1(1)255$. There is no introduction or description. The table is believed to have been computed on a Model 701 International Business Machine. The author believes that all digits are correct.

C. B. T.

**80[F].**—EMMA LEHMER, "On the cyclotomic numbers of order sixteen," *Canadian Jn. Math.*, v. 6, 1954, p. 449–454.

DICKSON proved that $(i, j)_8$, the number of solutions of

$$g^{8x+i} + 1 \equiv g^{8y+j}(p)$$

can be expressed for every $i, j$ as a linear combination, with rational coefficients whose denominators are divisible only by factors of 64, of $p, x, y, a, b$, where $p = x^2 + 4y^2 = a^2 + 2b^2$ and $p$ is a prime of the form $8n + 1$. The signs of $x, y, a, b$ have to be chosen in a special way which was discussed in another paper by the

author "On the number of solutions of $u^k + D \equiv w^2 \pmod{p}$," *Pac. J. Math.*, v. 5, 1955, p. 103–118.

In order to decide whether analogous relations hold for $(i, j)_{16}$, with coefficients whose denominators are divisible by factors of 256, computations were carried out on SWAC for a set of prime numbers of the form $32n + 1$ and $32n + 17$. These computations lead to the conclusion that Dickson's result cannot be extended to 16, at least not with the same sign rules which worked for $(i, j)_8$.

Some of the numerical results obtained for this investigation are published here as a table. It is possible to obtain 205 of the 256 values of $(i, j)_{16}$ from the 51 values of $(i, j)_{16}$ for which $i \leq j$, $j - i \geq i$, $16 - j > i$. Only these $(i, j)_{16}$ are tabulated for the eight primes of the form $32n + 1$:

$$97, 193, 449, 641, 673, 769, 929, 1409$$

and for the six primes of the form $32n + 17$:

$$241, 401, 433, 977, 1009, 1297.$$

Also the values of $x$, $y$, $a$, $b$ are given for each $p$, with the signs needed for the relations concerning $(i, j)_8$.

Printing error: p. 449, line 7 from top *should read*: $x \equiv a \equiv 1(4)$.

O. TAUSSKY

National Bureau of Standards
Washington, D. C.

**81[F].**—FRANCIS L. MIKSA, "A table of integral solutions of $a^2 + b^2 + c^2 = r^2$," *Mathematics Teacher*, v. 48, 1955, p. 251–255.

Solutions for $r = 3(2)207$. See Review 82 in this issue.

This paper lists 14 errata which the author has found in the earlier table by R. H. BACON [1], which listed solutions for $r = 3(2)99$.

C. B. T.

1. R. H. BACON, "Integral solutions of $x^2 + y^2 + z^2 = r^2$," *School Science and Mathematics*, v. 47, 1947, p. 155–164.

**82[F].**—FRANCIS L. MIKSA, *A table of integral solutions of $A^2 + B^2 + C^2 = R^2$ for all odd $R$ from $R = 3$ to $R = 325$, in two sets, $R = 3$ to 207 and an extension from $R = 207$ to $R = 325$.* 32 leaves, reproduced from typewritten manuscript. On deposit in UMT FILE.

This table extends earlier tables by R. H. BACON [1] and the author [2] to $R = 325$. Bacon's table listed decompositions of odd numbers into the sum of three squares for numbers $R = 3(2)99$, and the paper by MIKSA presented the list for $R = 3(2)207$. The calculation follows the lines of the two described; the present table has a clear introduction telling how it was calculated and listing a few pertinent identities. For $R \geq 209$ non-primitive sets are marked.

C. B. T.

1. R. H. BACON, "Integral solutions of $x^2 + y^2 + z^2 = r^2$, *School Science and Mathematics*, v. 47, 1947, p. 155–164.
2. FRANCIS L. MIKSA, "Table of integral solutions of $a^2 + b^2 + c^2 = r^2$," *Mathematics Teacher*, v. 48, 1955, p. 251–255.

**83[F].**—FRANCIS L. MIKSA, *Table of quadratic partitions* $x^2 + y^2 = N$. Manuscript of 102 leaves (reproduced from typewritten masters) deposited in the UMT FILE.

The table attempts a complete description of the possible decomposition of all odd numbers $N = 1(2)50{,}000$ into the sum of two squares. Some pertinent information concerning this is contained in [1]. Rules for deriving similar decomposition of even numbers are given in the introduction. No details of the calculation are included.

C. B. T.

1. J. V. USPENSKY & M. A. HEASLET, *Elementary Number Theory*, McGraw-Hill, New York, 1939, p. 337–341.

**84[F].**—R. J. PORTER, *Irregular Negative Determinants of Exponent 3n with their critical classes. Part I, from* $-D = 1$ *to* 50,000. 174 typewritten pages, 25.5 × 10 cm., deposited in UMT FILE.

This table is a supplement to *Table of irregular negative determinants of exponent 3n up to* $-2 = 150{,}000$, which were previously deposited in the UMT FILE [*MTAC*, v. 9, 1955, p. 26].

R. J. PORTER

266 Pickering Road
Hull, England

**85[F, I, J, K].**—FRANCIS L. MIKSA, *Stirling numbers of the second kind.* Typewritten manuscript (reproduced from typewritten masters) of 32 p., on deposit in UMT FILE.

This table gives ${}_rS_n$ for $n = 1(1)50$, a table of $G_n$ for $n = 1(1)51$, an excellent introduction, and a bibliography prepared by HENRY FINLAYSON. The number ${}_rS_n$ is the Stirling number of the second kind defined by the expansion formula. It is also given as

$$\frac{\Delta^r 0^n}{r!}.$$

The number $G_n = \sum_r {}_rS_n$.

The author gives the recurrence relations used in computing and in checking the tables. The numbers are used in number theory, some aspects of analysis dealing with series and with probability or distribution problems involving arrangements of $n$ objects.

Other published lists are reported by FLETCHER, MILLER and ROSENHEAD [1].

C. B. T.

1. FLETCHER, MILLER & ROSENHEAD, *An Index of Mathematical Tables*, Scientific Computing Service, Ltd., London, p. 72.

**86[H].**—ANDRES ZAVROTSKY, *Tablas para la Formacion de las Ecuaciones Cubicas*, Publicaciones de la Direccion de Cultura de la Universidad de los Andes, No. 44, Merida, Venezuela, 1955, iv + 48 pp., 21.5 × 15.8 cm.

This table gives the exact values of the coefficients of the cubic equation $x^3 + px + q = 0$ as a function of two of its roots $x_1$ and $x_2$ (the third root $x_3$ being $-x_1 - x_2$). The functions $-p$ and $q$ are tabulated side by side for $x_1 = 1(1)100$

and $x_2 = x_1(1)100$, where $-p = x_1^3 + x_2 + x_1x_2$ and $q = x_1x_2(x_1 + x_2)$. There is a brief introduction containing some notes on the early history of the subject and some obvious comments on how to obtain $p$ and $q$ from the table when $x_i$ is either negative or non-integral, which is followed by some dozen references to other tables for solving cubics.

HERBERT E. SALZER

Diamond Ordnance Fuze Laboratories
Washington, D. C.

87[H, I].—EDUARD L. STIEFEL, *Kernel polynomials in linear algebra and their numerical applications*, multilithed from typescript, 1955, 52 p., 26.7 × 20.3 cm., possibly available on loan from libraries or the author.

High-speed digital computing machines make it possible to deal with matrices of much larger orders than would have been considered fifteen years ago. As a consequence, in the last decade many older matrix methods have been re-examined, and a number of new methods—mostly iterative—for treating linear problems have been devised by FRAME, HESTENES, KANTOROVICH, LANCZOS, PURCELL, SOURIAU, STIEFEL, D. YOUNG, and others. The practical computer faces an almost impossible task of remembering all these methods for solving linear problems, not to mention understanding them or being able to predict which is best for a given task. The reviewer's outline [1] of methods of solving linear systems is terrifying by its length and lack of unity.

But now Professor Stiefel has unified several of the known iterative methods for matrix problems. The unifying principle is the theory of orthogonal polynomials over an arbitrary mass distribution on an interval of the real axis. The booklet reviewed here is a clear and careful exposition of this material.

In the first section the author reviews the theory of orthogonal polynomials $\{P_i(\lambda)\}$ over a mass distribution $\rho(\lambda)d\lambda$. He shows how one can generate them by orthogonalizing the sequence $\{\lambda P_{i-1}(\lambda)\}$, a far more efficient process than orthogonalizing the powers $\{\lambda^i\}$, as Lanczos has stressed. Although the author doesn't mention it here, this idea can be worked into a polynomial curve-fitting routine much more effective than the usual ones based on solving normal equations built from $\{\lambda^i\}$.

To solve a linear system $Ax = k$, the author considers simple iterative routines of the form $x_{i+1} = x_i + \Delta x_i$, where $\Delta x_i = (1/q_i)r_i$, $r_i = k - Ax_i$. Then $r_n = R_n(A)k$, where $R_n(\lambda) = \prod_{k=0}^{n-1} (1 - \lambda/q_k)$ is a polynomial with $R_n(0) = 1$. The author poses the following problem of choosing a best *strategy*: Suppose the eigenvalues of $A$ are known to lie in an interval $[a, b]$. Then it is desired to find that polynomial $R_n(\lambda)$ of maximal degree $n$ such that $R_n(0) = 1$ and such that $\int_a^b R_n(\lambda)^2\rho(\lambda)d\lambda$ is minimized. Since one doesn't know the eigenvalues $\lambda_i$, such a requirement is about all that one can make about the smallness of $R_n(\lambda_i)$. The author solves this minimization problem exactly in terms of orthogonal polynomials, and shows various ways in which it may be used to construct algorithms which yield small $r_n$. He points out, however, that the intermediate residuals $r_i$ may be quite large; i.e., these processes may be unstable.

The author now shows how to modify the above algorithm so that *each inter-*

*mediate $r_i$ also corresponds to a best strategy*, thus restoring stability. The change is to put $\Delta x_i = (1/q_i)(r_i + p_i \Delta x_{i-1})$, where $p_i$, $q_i$ are scalars involved in the three-term recurrence relation for $\{P_i(\lambda)\}$. He shows that various known algorithms correspond to special choices of $\rho(\lambda)$—in particular, the conjugate gradient method of Lanczos, Hestenes, and Stiefel.

In another section the same ideas are applied to the calculation of eigenvalues of a matrix $A$ with real eigenvalues. It is shown how to bring out any intermediate eigenvalue by iterations with an appropriate *kernel polynomial*. As a special case the author shows us the intriguing new *spectroscopic method* of Lanczos. The reviewer feels that the ideas of this booklet will prove more useful in eigenvalue problems than in solving linear systems, because of the greater number of other ways of dealing with linear systems.

The final section is devoted to an application of the *quotient-difference* algorithm of RUTISHAUSER and Stiefel to generating the orthogonal polynomials.

This booklet is an exceedingly valuable contribution, and it is hoped that the material will be made available to every mathematician concerned with numerical methods in linear algebra. A person studying the booklet carefully will see—probably for the first time—the unity behind the gradient methods, RICHARDSON's method, and the various conjugate gradient methods. And he will learn how to improve his techniques of curve fitting.

The author uses the notation of the Dirac delta function for atomic mass distributions, and also to motivate the derivation of the kernel polynomials. This is most unusual for a mathematician, and is probably done in an effort to reach non-mathematical readers. While the reviewer is neutral on the matter, he finds that several of his colleagues resent it emphatically. The reviewer's only criticism is that there is no direct reference to the work of L. F. Richardson, who in his 1910 paper [2] was considering similar questions of strategy. The reviewer also wonders whether the author couldn't as well assume at many places in the exposition that $A$ has complex eigenvalues, and thus broaden the treatment substantially. After all, a limitation to real eigenvalues is almost a limitation to symmetric matrices.

GEORGE E. FORSYTHE

University of California
Los Angeles, California

1. G. E. FORSYTHE, "Tentative classification of methods and bibliography on solving systems of linear equations," *National Bureau of Standards Applied Mathematics Series 29*, 1953, Superintendent of Documents, Washington, D. C., p. 1–28.
2. L. F. RICHARDSON, "The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam," Roy. Soc. London, *Phil. Trans.* (A), v. 210, 1910, p. 307–357.

Editorial note. These were notes of four lectures prepared under contract between the National Bureau of Standards and the American University under sponsorship of the Office of Naval Research. They will probably be published as part of a volume to appear within a few months as part of the NBS Applied Mathematics Series.

**88[H, K, P, S, Z].**—*Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* held at the University of Chicago, April 29–30, 1954. Vol. II. Editorial Committee: F. E. GRUBBS, F. J. MURRAY, & J. J. STOKER, Interscience Publishers, Inc., 1955, 216 p., $5.00.

This volume contains eleven papers based on invited addresses given by the separate authors at the second symposium on applied mathematics sponsored

jointly by the American Mathematical Society and the Office of Ordnance Research, U. S. Army. This symposium was held on April 29 and 30, 1954 at the University of Chicago. A great variety of different fields of applied mathematics and mathematics is displayed in this book, ranging over operational research, statistics, and partial differential equations.

P. M. MORSE discusses Operations Research as an effort to discover regularities in the behavior of teams of men and equipment as they go about doing assigned jobs and to link these regularities with other knowledge so that the phenomena occurring can be modified and controlled. Examples of the application of operations research to various problems are given.

J. NEYMAN gives a consistent presentation of his fundamental contributions to the theory of inductive inference and answers objections raised by CARNAP. Neyman propounds the view that prior to any discussion based on a theory of inductive inference a theorizing step must be taken to create a mathematical model of the phenomenon being studied. He points out that the model may differ from the theory of probability based on the axioms of Kolmogoroff. The general discussion of the first part of the paper is applied to the problem of homogeneity of neutral V-particles.

H. O. HARTLEY reports progress on two topics in his paper entitled, "Some Recent Developments in Analysis of Variance." These topics are: (1) multiple decisions and comparisons and (2) short cut procedures based on substitute measures of dispersion.

The "Two Unsolved Problems of Statistical Mechanics" explained by J. E. MAYER are:

(1) The evaluation of the pressure $P$ of a system with $N$ molecules as a function of the activity $Z$ and the temperature $T$ as a sum over integrals of functions of the potential energy over configurations in 3 $N$ dimensional space. The equivalence of this problem and the solution of a compact matrix equation is pointed out.

(2) The concise logical formulation of problems involving the rate of approach to equilibrium.

M. R. HESTENES in his paper, "Iterative Computational Methods," is concerned mainly with matrix problems and with problems that arise from variational principles. The main concern is with gradient methods and a discussion of the conjugate gradient method of solving linear systems of equations is included.

J. TODD's paper entitled "Motivation for Working in Numerical Analysis," contains a brief discussion of various separate topics in numerical analysis. These include evaluation of polynomials, increasing speed of convergence of sequences, modified differences, characteristic roots of finite matrices, quadrature, integral equations, and game theory. The author stresses the need for an experimental and empirical approach to many problems in numerical analysis.

In the paper, "Some Numerical Computations in Ordnance Problems," A. A. BENNETT lists and discusses briefly 40 problems considered by the Computation Laboratory of the Ballistic Research Laboratories, Aberdeen Proving Ground.

C. TRUESDELL in the paper, "The Simplest Rate Theory of Pure Elasticity," postulates a set of ten non-linear equations for the following ten quantities: The density, the three components of the velocity vector field, and the six components of the stress tensor. These equations embody the law that the rate of stress is a

function of the rate of deformation. Simple special solutions of the equations are discussed.

J. J. STOKER reviews various definitions of stability in his paper, "On the Stability of Mechanical Systems," and discusses briefly the possibility of converting inherently unstable situations into stable ones by applying periodic external forces.

The longest paper in the collection is the one by F. J. BUREAU entitled "Divergent Integrals and Partial Differential Equations." In this paper the author reviews the definition of the finite part and the logarithmic part of divergent integrals and uses these concepts to obtain the solutions to the Cauchy problem for the wave equation, the equation for damped wave and the Euler-Poisson-Darboux equation. Linear partial differential equations with constant coefficients and systems of such equations are also discussed. HADAMARD's method of singularities for obtaining solutions to such equations is reviewed and new applications of this method are described. The usefulness of the notions of principle and logarithmic part of divergent integrals in the theory of elliptic equations is illustrated by a discussion of a solution of RADON's problem. Other related problems are also treated.

A. H. T.

**89[1].**—E. W. DIJKSTRA & A. VAN WIJNGAARDEN, *Table of Everett's Interpolation Coefficients*, Computation Department of the Mathematical Centre, Amsterdam, Report R294, printed by Excelsior's Photo-Offset, The Hague, 1955. Introduction of 1 page plus 200 pages of tables (pages unnumbered), 24.7 cm.

The EVERETT interpolation formula for a function $f(x)$ that is tabulated at equal intervals of the argument $x_k = x_0 + kh$ is expressible as

$$f_p = (1-p)f_0 + pf_1 + E_0^2(p)\delta^2 f_0 + E_1^2(p)\delta^2 f_1 + E_0^4(p)\delta^4 f_0 + E_1^4(p)\delta^4 f_1 + E_0^6(p)\delta^6 f_0 + E_1^6(p)\delta^6 f_1 + \cdots,$$

where $f_p = f(x_0 + ph)$, $f_k = f(x_k)$, $\delta^{2n} f_k$ are the $(2n)^{\text{th}}$ central differences of $f_k$, and the interpolation coefficients $E_k^{2n}(p)$, $k = 0, 1$, are given by

$$E_k^{2n}(p) = (-1)^{k+1}\binom{p+n+k-1}{2n+1}.$$

These present tables provide the six functions $E_k^{2n}(p)$; $k = 0, 1$; $n = 1, 2, 3$; for $p = 0(0.0001)1$; to 7D. Use could have been made of the symmetry relation $E_0^{2n}(p) = E_1^{2n}(1-p)$ to reduce the size of the book by 50%, either by giving $E_0^{2n}(p)$ or $E_1^{2n}(p)$ alone for $p = 0(0.0001)1$ or by giving both functions only as far as $p = 0.5000$. But this present arrangement saves the reader the extra effort of looking up the coefficients in two different parts of the book, and it also reduces the chances of error. All six functions $E_k^{2n}(p)$ are tabulated alongside of each other. There are 51 arguments $p$ on each page, in groups of five except for the last argument on the page which is repeated at the top of the next page. The fine interval of 0.0001 in $p$ will, in many cases, obviate the need for interpolating in this table.

These present tables might be compared with two [1] earlier works each

containing extensive tabulations of Everett coefficients. A. J. THOMPSON's excellent book includes, among other shorter basic tables of $E_k^{2n}(p)$, a main table of $E_k^{2n}(p)$ as far as $2n = 8$, for $p$ at intervals of 0.001, with central differences to aid in the interpolation for $E_k^{2n}(p)$ when $p$ occurs to more than 3D. The NATIONAL BUREAU OF STANDARDS volume includes $E_0^2(p)$ and $E_1^2(p)$ at intervals of 0.0001 and $E_0^4(p)$, $E_1^4(p)$, $E_0^6(p)$, $E_1^6(p)$ at intervals of 0.001, all to 10D. In the $A$-notation of this table, $E_0^{2n}(p) = A_{n+1}^{2n+2}(1 - p)$, $E_1^{2n}(p) = A_{n+1}^{2n+2}(p)$. However, the user of the Lagrangian table would have the extra work of looking up the desired values of $E_k^{2n}(p)$ in different parts of the book. Thus in comparison with earlier tables this present work has the advantages of finer intervals and more convenient arrangement, although it does not have as many decimal places.

The table was produced on the electronic computer ARRA of the Mathematisch Centrum in Amsterdam. The program included complete arithmetic checks which also covered the signals transmitted to the typewriter. The final sheets were reproduced by photo-offset.

The tables were computed for the "Empresa nacional bazan de construcciones navales militares' of Madrid."

Misprints in the Introduction: line 3, *for* $2k$, *read* $2n$; line 14, *for* $E_0^{2k}(p) = E_1^{2k}(1 - p)$, *read* $E_0^{2n}(p) = E_1^{2n}(1 - p)$.

HERBERT E. SALZER

Diamond Ordnance Fuze Laboratories
Washington 25, D. C.

1. a) A. J. THOMPSON, *Tables of the Coefficients of Everett's Central-Difference Interpolation Formula. Tracts for Computers, No. V*, 2nd ed., Cambridge Univ. Press, 1943.
b) NATIONAL BUREAU OF STANDARDS, *Tables of Lagrangian Interpolation Coefficients*, Columbia Univ. Press, New York, 1944.

**90[I].**—OWEN R. MOCK, *Fifty Everett integration coefficients*. Three typewritten leaves (on vellum), deposited in UMT FILE.

The first fifty-one coefficients to use in EVERETT's integration formula

$$\int_{x_0}^{x_1} y(x)dx = \tfrac{1}{2} \sum_{r=0}^{\infty} M_{2r}\delta^{2r}[y(x_0) + y(x_1)],$$

listed to 50D and believed by the author to be accurate to 48D. There is no introduction or description; the formula is discussed by MILNE [1]. The list is believed to have been computed on a Model 701 International Business Machine.

C. B. T.

1. W. E. MILNE, *Numerical Calculus*, Princeton, 1949, p. 198.

**91[I].**—H. E. SALZER, "A simple method for summing certain slowly convergent series," *J. Math. Physics*, 33, 1955, p. 356–359.

This gives a table of (the integers)

$$B_{n,n-i}^{(m)}, \quad D_n^{(m)}, \quad \text{where} \quad A_{n,n-i}^{(m)} = \frac{(-1)^i(n-i)^{m-1}}{(m-1)_7}\binom{m-1}{i} = B_{n,n-i}^{(m)}/D_n^m$$

and where $D_n^{(m)}$ is the least common denominator of the $A_{n,n-i}^{(m)}$, $i = 0, 1, \cdots, m - 1$. The following values are covered: $n = 5$, $m = 4$; $n = 10$, $m = 4$, $m = 7$;

$n = 15$, $m = 4, 7, 11$; $n = 20$, $m = 4, 7, 11$. The motivation for this tabulation is the following: Consider the partial sums $s_r$ of a convergent series as a function $s(x)$ of $x = r^{-1}$. The problem of the determination of $s = \lim s_n$ is regarded as that of interpolation (or rather extrapolation), for $s(0)$ given $s(p^{-1})$ for some integral values of $p$, in particular $p = n, n-1, \cdots, n-m+1$. This is solved by Lagrangian interpolation and

$$s \doteq \sum_{i=0}^{m-1} A_{n,n-i}^{(m)} s_{n-i}.$$

Various standard examples are discussed. The method appears effective when the sequence $\{s_r\}$ is monotone.

J. T.

**92[I].**—H. E. SALZER, "Equally weighted quadrature formulas over semi-infinite intervals," *J. Math. Physics*, 34, 1955, p. 54–63.

In general it is not possible to find real $x_i = x_i^{(n)}$ such that the quadrature formulae

$$\int_0^\infty e^{-x} f(x)dx = 1/n \sum_{i=1}^{n} f(x_i) \tag{1}$$

or

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x)dx = \frac{\pi^{\frac{1}{2}}}{n} \sum_{i=1}^{n} f(x_i) \tag{2}$$

are exact when $f(x)$ is a polynomial of degree $n$. The polynomials having the $x_i^{(n)}$ for zeros are obtained formally for $n = 2(1)10$ in both cases [cf. L. M. MILNE-THOMSON, *Calculus of Finite Differences*, London, 1933, p. 177–180]. Among these only that for $n = 2$ in (1) and those for $n = 2, 3$ in (2) have all roots real. The roots are, however, given in all cases to 7D. There follows a detailed discussion of a few quadrature formulae of the above type which require exactitude only for polynomials of degree $m < n$, e.g., a seven-point quadrature which is exact for cubics in case (1) and a seven-point quadrature which is exact for quintics in case (2).

J. T.

**93[I].**—HERBERT E. SALZER, *Formulas for calculating Fourier coefficients*, 5 typewritten pages (photocopy) placed in the UMT FILE.

These present formulas enable one to calculate the first twenty-four Fourier cosine and sine coefficients of a function $f(x)$ from its values (either given, or obtained through calculation or interpolation in a table of $f(x)$ at nine equally spaced points ranging from 0 to $2\pi$). The coefficients $a_i$, $i = 1(1)5$, and $b_i$, $i = 1(1)4$, are tabulated for $n = 0(1)24$, to five decimal accuracy, for use in these formulas:

$$(1)\quad \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx\,dx = a_1[f(0) + f(2\pi)] + a_2\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{7\pi}{4}\right)\right] + a_3\left[f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{2}\right)\right] + a_4\left[f\left(\frac{3\pi}{4}\right) + f\left(\frac{5\pi}{4}\right)\right] + a_5 f(\pi),$$

and

$$(2)\quad \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx = b_1[f(0) - f(2\pi)] + b_2\left[f\left(\frac{\pi}{4}\right) - f\left(\frac{7\pi}{4}\right)\right] + b_3\left[f\left(\frac{\pi}{2}\right) - f\left(\frac{3\pi}{2}\right)\right] + b_4\left[f\left(\frac{3\pi}{4}\right) - f\left(\frac{5\pi}{4}\right)\right].$$

Apart from the error due to the approximate nature of $a_i$, $b_i$, and $f\left(\frac{m\pi}{4}\right)$, formulas (1) and (2) are exact whenever $f(x)$ is any polynomial of degree $\leq 8$.

HERBERT E. SALZER

Diamond Ordnance Fuze Laboratories
Washington 25, D. C.

**94[I].**—HERBERT E. SALZER, *Formulas for inverse osculatory interpolation*, 5 typewritten pages (ozalid) placed in the UMT FILE.

The author has previously given inverse interpolation formulas [1], [2] for finding $x = x_0 + ph$ from $f(x)$, in terms of $f_i \equiv f(x_i)$, where the $x_i \equiv x_0 + ih$ are equally spaced at intervals of $h$. Those formulas were obtained by the inversion of LAGRANGE's interpolation formula.

The formulas give $p$ in terms of $f(x_0 + ph) \equiv f$, $f(x_i) \equiv f_i$, and $f'(x_i) \equiv f'$ for $x_i \equiv x_0 + ih$ at equally spaced intervals $h$, where the $i$ ranges from $-[(n-1)/2]$ to $[n/2]$, for $n = 2(1)7$, where $n$ is the number of points required in direct osculatory interpolation. Although the direct interpolation formula for $n = 6$ and $n = 7$ is of the 11th and 13th degree accuracy respectively, the inversion formula for $p$ which is given does not go beyond the tenth degree terms (and in most practical problems it is very rarely that one will go that far; in fact, the first few terms will usually suffice).

HERBERT E. SALZER

Diamond Ordnance Fuze Laboratories
Washington, D. C.

1. H. E. SALZER, "A new formula for inverse interpolation," Amer. Math. Soc., *Bull.*, v. 50, 1944, p. 513–516.
2. H. E. SALZER, "Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation," *J. Math. and Phys.*, v. 24, 1945, p. 106–108.

**95[K].**—E. S. PEARSON & H. O. HARTLEY, *Biometrika Tables for Statisticians*, v. 1, Cambridge University Press, 1954, xiv + 238 p. 26.0 cm. Price, $4.50.

One of the objectives of the originators of the journal *Biometrika* was to provide a place where mathematical tables of importance in statistics could be published. A number of such tables were collated to form the volume I of *Tables for Statisticians and Biometricians* which appeared for the first time in 1914. Four subsequent editions were issued and a second volume appeared in 1931. Due to the phenomenal growth of new developments in statistical methods since that time, it was felt that a modern version was warranted and thus the present *Biometrika* tables. This is the first volume of two and contains fifty-four of the more commonly used tables. The second volume will be concerned with the presentation of more specialized functions. The arrangement of the tables is such as to

provide convenience to the user, the grouping being to collect the tabulations of individual or related functions. The sources from which the tables were drawn are given in an appendix. The older tables, having been in use for a considerable time, can be used with the satisfaction of knowing that they are error free. For the tables that have been newly compiled one can be certain that painstaking efforts, customary from English computers, have been made to guarantee their accuracy.

An outstanding feature of the volume is the comprehensive introduction prefacing the tables. Here the underlying mathematical theory is discussed, mathematical definitions are provided and a careful exposition is given of the method of interpolation to be used in each table, illustrated by example. In addition, the usefulness of each table is stressed by the presence of typical applications. Teachers of mathematical statistics should find this introduction an important supplement to their classroom texts. Because of the excellence of the introduction one may consider the volume as a book in mathematical statistics supplemented by tables rather than a book of tables preceded by an introduction.

The tables have been grouped under six main headings. The first group contains tables of the normal probability function:

1. The integral $P(X)$ and the ordinate $Z(X)$ in standard units to 7D for $X = 0(.01)4.5$ and to 10D for $X = 4.5(.01)6$ with first and second central differences. This is the fundamental table due to W. F. SHEPPARD.
2. Values of $-\log Q(X) = -\log [1 - P(X)]$ to 5D for $X = 5(1)50(10)100(50)500$.
3. Values of $X$ for extreme values of $Q$ and $P$ to 4D for $Q = 0(.0001).02$.
4. Values of $X$ in terms of $Q$ and $P$ to 4D for $Q = 0(.001).5$.
5. Values of $Z$ in terms of $Q$ and $P$ to 5D for $Q = 0(.001).5$.
6. Table for probit analysis. Tabulates $P$, $Y + \frac{Q}{Z}$, $Y - \frac{Q}{Z}$, $\frac{1}{Z}$, and $\frac{Z^2}{PQ}$ for the expected probit $Y = 1(.01)9$. $P$ is given to 3D for $Y = .5(.01)6.5$, to 4D for $Y = 6.5(.01)7.5$, to 5D for $Y = 7.5(.01)8.5$, and to 6D for $Y = 8.5(.01)9$. The other functions are given to 4D.

The second group is concerned with tables derived from the normal probability function:

7. The probability integral of the $\chi^2$ distribution and the cumulative sum of the Poisson distribution. $Q(\chi^2/\nu)$ is given to 5D for $\chi^2 = .001(.001).01(.01).1(.1)2(.2)10(.5)20(1)40(2)134$ and $\nu = 1(1)\cdots$.
8. Values of $\chi^2$ to 5 to 7S (mostly 6S) for $Q(\chi^2/\nu) = .995, .99, .975, .95, .9, .75, .5, .25, .1, .05, .025, .01, .005, .001$ and $\nu = 1(1)30(10)100$.
9. The probability integral of the $t$-distribution, $P(t|\nu)$ to 5D for $t = 0(.1)8$ and $\nu = 1(1)24, 30, 40, 60, 120, \infty$. Upper percentage points of $t$ for $1 - P(t|\nu) = .001, .0001, .00001, .000005$ and $\nu = 1(1)10$ are given to 3 or more S.
10. Chart for determining the power function of the $t$-test. The probability $\beta$ of establishing significance using the two-tail test at the $\alpha = 5\%$ or $1\%$ level for values $\varphi = E(y)/\sqrt{2}\sigma_y$ is shown on a logarithmic scale. For $\alpha = .01$,

$\varphi$ ranges from 2 to 5; for $\alpha = .05$ from 1.2 to 3.5 for the degrees of freedom $\nu = 6(1)10, 12, 15, 20, 30, 60, \infty$, $\beta$ has the range .1 to .99.

11. Test for comparisons involving two variances which must be separately estimated. The table gives critical values to 2D of the ratio $v = (y - \eta)/(\lambda_1 s_1^2 + \lambda_2 s_2^2)^{\frac{1}{2}}$ for the significance level $\alpha = .05$ and degrees of freedom, $\nu_1$, $\nu_2 = 6, 8, 10, 15, 20, \infty$; for $\alpha = .01$, $\nu_1$, $\nu_2 = 10, 12, 15, 20, 30, \infty$ and for $c = \lambda_1 s_1^2/(\lambda_1 s_1^2 + \lambda_2 s_2^2) = 0(.1)1$.
12. Percentage points of the $t$-distribution. For values of $Q = 1 - P(t|\nu) = .4, .25, .1, .05, .025, .01, .005, .0025, .001, .0005$, and $\nu = 1(1)30, 40, 60, 120$ and $\infty$, values of the percentage points are given to 3D.
13. Percentage points for the distribution of the correlation coefficient, $r$, when $\rho = 0$. The values to 3D are given for $Q = .05, .025, .01, .005, .0025$, and .0005 for the degrees of freedom, $\nu = 1(1)20(5)50(10)100$.
14. Values of $z = \tanh^{-1} r$, with proportional parts are given to 4D for $z < 1$ and 3D for $z > 1$ for $r = .000(.002).998$.
15. Charts giving confidence limits for the population correlation coefficient $\rho$, given the sample coefficient, $r$ with confidence coefficients .95 and .99. Values of $r$ and $s$ are given from $-1.(.05)1$ for sample sizes $= 3(1)8(2)12, 20, 25, 50, 100, 200$ and 400.
16. Percentage points of the B-distribution. Values of $x$ are given to 5S for which

$$I_x(a, b) = \frac{\int_0^x u^{a-1}(1-u)^{b-1}du}{\int_0^1 u^{a-1}(1-u)^{b-1}du} = .5, .25, .1, .05, .025, .01 \text{ and } .005,$$

where $a = \frac{1}{2}\nu_2$, $b = \frac{1}{2}\nu_1$ for $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$ and $\nu_2 = 1(1)30, 40, 60, 120, \infty$.

17. Chart for determining the probability levels of the incomplete $B$-function, $I_x(a, b)$. Three separate charts are given: Chart A for $1 \leq b \leq 4$, Chart B for $4 \leq b \leq 15$ and Chart C for $15 \leq b \leq 60$. Each chart is further divided into upper and lower parts. The lower part contains a family of curves in Chart A corresponding to values of $a = 2(1)10(2.5)25(5)50(10)100(25)150, 200$. The lower parts of Charts B and C are given at the same intervals except that the starting values are $a = 5$ and 20 respectively. The top part contains a family of curves corresponding to selected values of $I_x(a, b)$. In charts A and B, $I_x(a, b) = .005, .01, .02, .05, .1(.1).9, .95, .98, .99, .995, .999, .9995$. In chart $B$, in addition, the curve corresponding to the value .9999 is also given. In chart C, $I_x(a, b) = .0005, .001, .005, .01, .025, .05, .1(.1).9, .95, .975, .990, .995, .999, .9995$.
18. Percentage points of the $F$-distribution (variance ratio). These are given for degrees of freedom, $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$, $\nu_2 = 1(1)30, 40, 60, 120, \infty$, for the upper 25, 10, 5, 2.5, 1, .5 and .1 percentage points to 2D if $F < 100$, 1D if $F < 1000$, to the nearest integer if $F < 30{,}000$ and to 4S for $F > 30{,}000$.
19. Percentage points of the largest variance ratio $s^2_{\max}/s_e^2$. Values of $s^2_{\max}/s_e^2$ for the upper 5 and 1% points are given to 2D for the degrees of freedom for

$s_o^2$, $v = 10, 12, 15, 20, 30, 60, \infty$, for $k = 1(1)10$. The quantity $k$ here is the number of independent variance estimates each based on 1 degree of freedom of which $s^2_{max}$ is the largest.

Group three comprises further tables of probability integrals, percentage points, etc., of distributions derived from the normal function.

20. Moment constants of the mean deviation and of the range. For the mean deviation this table gives for $n = 2(1)20, 30, 60$ the expectation to 6D, the standard deviation to 4D, variance to 5D and first and second moment ratios to 3 or 4D. For the range and the same values of $n$, values of the expectation to 5D, standard deviation to 4D, and variance of the range to 5D, are given with the first and second moment ratios to 4 and 3D respectively. The factors $d_n/V_n$ and $d_n^2/V_n$ where $d_n$ and $V_n$ are the expectation and variance are given to 3S.
21. Percentage points of the distribution of the mean deviation. The lower and upper .1, .5, 1, 2.5, 5 and 10 percentage points of the standardized mean deviation as obtained from the probability integral table are given for sample size $n = 2(1)10$ to 3D.
22. Percentage points of the distribution of the range. The same lower and upper percentage points as in Table 21 of the distribution of the range are given to 2D for $n = 2(1)20$.
23. Probability integral of the range $W$, in normal samples of size $n$. The values of $P(W/n)$ are tabulated to 4D for $n = 2(1)20$ and $W = 0.00(.05)7.25$.
24. Percentage points of the extreme standardized deviate from population mean $(x_n - \mu)/\sigma$ or $(\mu - x_1)/\sigma$. The lower and upper .1, .5, 1, 2.5, 5, 10% points of $X_n = (x_n - \mu)/\sigma$ are given for $n = 1(1)30$ to 3D.
25. Percentage points of the extreme standardized deviate from sample mean $(x_n - \bar{x})/\sigma$ or $(\bar{x} - x_1)/\sigma$. The lower and upper .1, .5, 1, 2.5, 5, 10% points of $u = (x_n - \bar{x})/\sigma$ are given for $n = 1(1)9$ to 2D.
26. Percentage points of the extreme studentized deviate from sample mean $(x_n - \bar{x})/s_\nu$ or $(\bar{x} - x_1)s_\nu$. Lower and upper 1 and 5% points of the studentized extreme deviate $t_n = (x_n - \bar{x})s_\nu$ where $s_\nu$ is an estimate of $\sigma$ based on $\nu$ degrees of freedom are given for $n = 3(1)9$ and $v = 10, 15, 30, \infty$ for the lower percentage points and $= 10(1)20, 24, 30, 40, 60, 120, \infty$ for the upper ones.
27. Mean range in normal samples of size $n$. Values of the expected value, $W_n$, in samples from a normal population are given to 5D for $n = 2(1)500(10)1000$.
28. Mean positions of ranked normal deviates (normal order statistics). Values of $\xi(i|n)$, the expectation of the $i$-th ranked normal deviate for $n = 2(1)26(2)50$, $i = 1(1)25$ to 3D for $n \leq 20$; to 2D for $n > 20$.
29. Percentage points of the studentized range $q = (x_n - x_1)/s_\nu$. Lower and upper 1 and 5% points of $q$ for $n = 2(1)20$ and $\nu = 10(1)20, 24, 30, 40, 60, 120, \infty$ for lower percentage points and $= 1(1)20, 24, 30, 40, 60, 120, \infty$ for the upper ones.
30. Tables for analysis of variance based on range. In part A the scale factor, $c$, to 2D and the equivalent degrees of freedom, $\nu$, to 1D appropriate to a simple classification into $K$ groups of $n$ observations are given for $n = 2(1)10$ and

$K = 1(1)5, 10, \infty$, together with constant differences. In part B the like values for analysis of double classification with $K$ blocks and $n$ treatments for $n = 2(1)9$ and $K = 2(1)10, 20, \infty$.

31. Percentage points of the ratio $s^2_{max}/s^2_{min}$ in a set of $K$ independent mean squares. For the upper 5% points, values are given to 3S; for the upper 1% points to either 2, 3 or 4S for $\nu = 2(1)10, 12, 15, 20, 30, 60, \infty$ and $k = 2(1)12$.
32. Test for heterogeneity of variance: percentage points of Bartlett's test, $M$. The 5% and 1% significant points of $M$ are given to 2D for $k = 3(1)15$ and $c_1 = 0(.5)5(1)10(2)14$. Corresponding to each combination of $k$ and $c_1$, two percentage points denoted by (a) and (b) are given, representing the maximum and minimum values of the true percentage point.
33. Test for heterogeneity of variance: table to facilitate interpolation in Table 32. This table required only for accurate interpolation in Table 32, gives $C = c_1^2/k^2$ and $\Delta C = c_1 - c^2/k^2$ to 3D for all the marginal entries $k$ and $c_1$ of Table 32.
34. Test for departure from normality.
    A. Percentage points of the distribution of $a$ = (mean deviation)/(standard deviation)
    B. Percentage points of the distribution of $\sqrt{b_1} = m_3/m_2^{\frac{3}{2}}$
    C. Percentage points of the distribution of $b_2 = m_4/m_2^2$

    Part A gives the upper and lower 10, 5 and 1% points for $a$, to 4D, the expectation and standard deviation to 5D for size of sample $n = 11(5)51(10)101(100)1001$. Part B gives the upper and lower 5 and 1% points for $\sqrt{b_1}$ to 3D and the standard deviation to 4D for size of sample $n = 25(5)50(10)100(25)200(50)1000(200)2000(500)5000$. Part C gives the upper and lower 5 and 1% points for $b_2$ to 2D for $n = 200(50)1000(200)2000(500)5000$.
35. Moments of $s/\sigma = \chi/\sqrt{\nu}$ and factors for determining confidence limits for $\sigma$. Values of the expectation are given to 6D of the standard deviation to 5D and of the standard deviation multiplied by $2\sqrt{\nu}$ to 4D. Values of the moment ratios $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ of $s/\sigma$ are given to either 4 or 5D for $\nu$ (degrees of freedom) $= 1(1)20(5)100$. In addition, the factors $\sqrt{\nu}/\chi_\alpha$ and $\sqrt{\nu}/\chi_{1-\alpha}$ for confidence limits of $\sigma$ are given for $\alpha = .025$ and .005 to 3 and 2D respectively.

The fourth section gives tables related to various discrete distributions.

36. Test for the significance of the difference between Poisson variables. In Part A the values of $b$ are given for which

$$\sum_{i=0}^{b} \binom{r}{i} \left(\tfrac{1}{2}\right)^r \leqq a \quad \text{and} \quad \sum_{i=0}^{b+1} \binom{r}{i} \left(\tfrac{1}{2}\right)^r > a$$

    for $\alpha$ (the nominal significance level) $= .1, .05, .025, .01, .005$ and $r = a + b$ in the range $1(1)80$. Part B for the single-tail test shows for values of $m = 2.5(2.5)25.0$ and the same nominal significance levels as in Part A, the probability of rejecting the hypothesis $m_1 = m_2$ when it is true. The probabilities are given to 3D for $\alpha = .1, .05, .025$ and to 4D for $\alpha = .01, .005$.

37. Individual terms of certain binomial distributions:

$$f(i|n, p) = \binom{n}{i} p^i (1 - p)^{n-i}$$

Values of $f(i|n, p)$ are given to 5D for $n = 5, i = 0(1)5$; $n = 10, i = 0(1)10$; $n = 20, i = 0(1)20$; $n = 25, i = 0(1)23$; $n = 30, i = 0(1)26$ and $p = .01, .02(.02).1(.1).5$.

38. Significance tests in a $2 \times 2$ contingency table. The table shows (1) in bold type for given $a$, $A$ and $B$, the value of $b(<a)$ which is just significant at the probability levels of 5, 2.5, 1 and $\frac{1}{2}\%$ for a single-tail test and (2) in small type for the given $A$, $B$ and $r = a + b$, the exact probability to 3D that $b$ is equal to or less than the bold type integer. $A$ covers the range from $3(1)15$; $B$ from $2(1)A$, and $a$ from $A(-1)$ until $b = 0$.
39. Individual terms, $e^{-m}m^i/i!$, of the Poisson distribution. Values are given to 6D for $m = .1(.1)15.0$ and $i$ ranging from $0(1)$ until the contribution becomes negligible in the 6th decimal place.
40. Confidence limits for the expectation of a Poisson variable. Tabulates the Poisson variable $m$ for the lower and upper .001, .005, .01, .025, .05 confidence limits to 3 or 4S for frequencies $c = 0(1)50$.
41. Charts providing confidence limits for $p$ in binomial sampling given a sample fraction $c/n$. Confidence coefficients, .95 and .99. These are charts of the confidence belts with the confidence limit $P$ as ordinate and the ratio $c/n$ as abscissa for $n$ ranging from 8 to 1000.

Miscellaneous tables such as the PEARSON type curves, rank correlation, and orthogonal polynomials are given in the fifth section.

42. Percentage points of Pearson curves, for given $\beta_1, \beta_2$, expressed in standardized measure. Tabulates the standardized deviate $X = \dfrac{x - \mu}{\sigma}$ to 3S for $\beta_1 = 0, .01, .03, .05, .1, .15, .2(.1)1$, and $\beta_2 = 1.8(.2)5$ for $P = .005, .01, .025, .05, .95, .975, .99, .995$.
43. Chart relating the type of Pearson frequency curve to the values of $\beta_1, \beta_2$. This chart drawn with $\beta_1$ and $\beta_2$ as coordinates in a rectangular coordinate system associates various types of Pearson curves with points in the $\beta_1\beta_2$-plane and indicates the significantly important regions of application.
44. Distribution of SPEARMAN's rank correlation coefficients, $r_s$, in random rankings. This table gives the probability $Q$ to 3 or 4D of exceeding or equalling a given score $S_r$. Results are given for $n = 4(1)10$, $S_r = 12(2)308$.
45. Distribution of KENDALL's rank correlation coefficient $t_k$, in random rankings. This table gives the probability $Q$ to 3 or 4D of exceeding or equalling a given score for $n = 4(1)9$, $S = 0(2)30$; $n = 6, 7, 10$, $S = 1(1)35$.
46. Distribution of the concordance coefficient $W$, in random rankings. Exact values of the probability $Q$ that a given score will be attained or exceeded are given for $n = 3$, $m = 3(1)10$; $n = 4$, $m = 3(1)6$; $n = 5$, $m = 3$.
47. Orthogonal polynomials. This table gives values of orthogonal polynomials $\phi_i(x)$ for $i = 1(1)6$ for sample sizes $n = 3(1)52$. The observational range covered is $t = 1(1)n$ for $n = 3(1)12$ and the half range $t = 1(1)[(n + 1/2)]$ for $n > 12$.

The final section gives various auxiliary tables which are of fundamental importance.

48. Powers of integers. This table gives the first $k$ powers of the integer $n$, for $n = 1(1)100$, $k = 2(1)7$.
49. Sums of powers of integers. This table gives the sums of the powers of the integers, $S(n^k)$ for $n = 1(1)100$, $k = 1(1)7$.
50. Squares of integers. Values of $n^2$ are given for $n = 0(1)999$.
51. Factorials of integers, their logarithms; square roots; and their reciprocals. For $n = 1(1)100$, $n!$ and $1/n!$ are given to 6S, while $\log_{10} n!$, $\sqrt{n}$, $1/\sqrt{n}$, and $1/n$ are given to 7D. For $n = 101(1)250$, $n!$ is given to 6S and $\log_{10} n!$ to 7D. For $n = 251(1)1000$, 7D are given of only $\log_{10} n!$
52. Miscellaneous functions of $p$ and $q = 1 - p$ over the unit range. For $p = .00(.01)1.00$, $\Gamma(1 + p)$ and $\log_{10}\{\Gamma(1 + p)\}$ are tabulated to 7D. The quantities $1 - p^2$, $pq$ and $p^2 + q^2$ are tabulated to 4D while $\sqrt{1 - p^2}1$, $\sqrt{1 - p^2}$ and $\sqrt{pq}$ are tabulated to 5D. For $p = .910(.002).990(.001)1.000$, the quantities $\sqrt{1 - p^2}1/\sqrt{1 - p^2}$ and $\sqrt{pq}$ are given to 5D.
53. Natural logarithms, $\log_e x$. $\text{Log}_e x$ is given to 5D together with proportional parts for $x = 1.00(.05)2.00(.01)9.99$.
54. Useful constants. Mathematical constants, their reciprocals and logarithms are given to 10D. Among these constants are multiples and powers of $\pi$, $e$, $e^2$, $\log_{10} e$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{10}$ and 1 radian. The binomial coefficients $\binom{n}{i}$ are given for $n = 4(1)12$ and $i = 0(1)12$. Conversion tables from metric to British and British to metric are given for measurements of length, area, volume, weight and capacity.

MILTON ABRAMOWITZ

National Bureau of Standards
Washington 25, D. C.

96[K].—F. W. DAVID & N. L. JOHNSON, "Statistical treatment of censored data. Part I. Fundamental formulae," *Biometrika*, v. 41, 1954, p. 228–240.

This paper is concerned with censored samples in which the smallest $k$ of $n$ observations are fully measured, while of the $(n - k)$ largest observations, it is known only that $x > x_k$. The authors consider a population described by a continuous random variable with p.d.f., $f(t)$, and with distribution function, $F(X) = \int_{-\infty}^{X} f(t)dt$. Arranged in ascending order of magnitude, observations of a (complete) random sample from a population of this type correspond to $x_1, x_2, \cdots, x_n$. Variables $X_1, X_2, \cdots, X_n$, defined by the equations $F(X_r) = r/(n + 1)$, $(r = 1, 2, \cdots, n)$, are introduced and approximations to the moments of the $x_r$'s are found by expanding $x_r$ about $X_r$ in an inverse TAYLOR series. Central moments and cumulants are then obtained from these noncentral moments. With the first four moments or cumulants known, functions of the PEARSON system are available for approximating distributions of the $x_r$'s and functions of them.

The first four orders of cumulants of $F(x_r)$ are listed in Table 1 which is appended to this paper. The first four orders of cumulants of the ordered variables $x_1, x_2, \cdots, x_n$ are listed in Table 2 with an accuracy of $0[(n + 2)^{-3}]$. Sums of powers and products of the first $k$ natural numbers which enter into calculation of the cumulants of Tables 1, and 2, are given in Table 3.

The application of results obtained in the paper is illustrated by determining upper and lower five percentage points of $\bar{x}_k$, which is the mean of the first $k$ observations (in ascending order of magnitude) of a censored sample from a normal population. Calculations are given to 3D for $n = 10$, $k = 2(2)8$ and for $n = 20$, $k = 4(4)16$.

A. C. COHEN, JR.

University of Georgia
Athens, Ga.

**97[K].**—F. G. FOSTER & A. STUART, "Distribution-free tests in time-series based on the breaking of records," Roy. Stat. Soc., *Jn.*, s.B., v. 16, 1954, p. 1–22.

Two distribution free tests making use of the statistics involving the upper records and the lower records for the randomness of a series are introduced in this paper. An observation in a time series is called a lower (upper) record if it is smaller (greater) than all previous observations in the series. The statistics used are $s$, the sum of the number of upper and lower records in a series and $d$, the difference between the number of upper and lower records. The hypothesis of randomness of a series can be tested by the critical region $s \geq s_0$ or $|d| \geq d_0$. The first test provides a test against trend in dispersion while the second tests against a trend in location. Comparable round-trip tests (S and D) are also introduced involving counting the records in the series in both directions.

The joint distribution of $s$ and $d$ is obtained under the null hypothesis and the two variables are shown to be independently distributed as well as being asymptotically normal. In the round trip case the corresponding statistics are established to be asymptotically normally distributed and the variances have been tabulated.

The powers of the tests against trend in the mean have been computed using empirical sampling techniques on a high speed computer.

The following tables are given:

Table 1. $P(s \leq s_0)$ to 3 or 4D for $n = 3(1)15$ and the normal approximation for $n = 15$.

Table 2. $P(d \leq d_0)$ to 3 or 4D for $n = 3(1)6$ with normal approximation for $n = 6$.

Table 3. Mean and standard error of $s$ and standard error of $d$ to 3D for $n = 10(5)100$ with approximation for $n = 100$.

Table 4. The standard error of D to 2D based on 1000 samples of size $n$ for $n = 10, 25, (25), 125$.

Table 5. The correlation between $d$ and $d'$ to 2D for $n = 10, 25(25)125$.

Table 6. The power of the $d$ test against normal regression at the 5 percent level for trends $\Delta = .01(.01).07$ and sample sizes $n = 25(25)125$ based on 1000 samples in each case.

Table 7. The power of the D test against normal regression at the 5 percent level for trends $\Delta = .01(.01).06, .08$, and sample size $n = 25(25)125$, based on 1000 samples in each case.

C. F. KOSSACK

Purdue University
Lafayette, Indiana

98[K].—H. E. DANIELS, "A distribution-free test for regression parameters," *Ann. Math. Stat.*, v. 25, 1954, p. 499–513.

Consider the usual regression model with $y_i = \alpha_0 + \beta_0 x_i + \epsilon_i$, where we assume only that the $x_i$ are distinct and that the errors $\epsilon_i$ are independent and satisfy Pr $(\epsilon_i > 0)$ = Pr $(\epsilon_i < 0) = \frac{1}{2}$. The lines $\alpha = -x_i\beta + y_i$ divide the $(\beta, \alpha)$ plane into polygonal regions, some of them open. Let $m$ denote the minimum number of lines which must be crossed to escape from $(\beta_0, \alpha_0)$ into an open polygon. The distribution of $m$ is given to 3D for $n = 3(1)30$, providing confidence regions for $(\beta_0, \alpha_0)$ and tests for an hypothesis specifying both parameters. The power of the $m$-test is compared with certain competitors, and the complications arising when not all $x_i$ are distinct are examined.

J. L. HODGES, JR.

University of California
Berkeley, California

99[K].—BENJAMIN EPSTEIN, "Tables for the distribution of the number of exceedances," *Ann. Math. Stat.*, v. 25, 1954, p. 762–768.

Two random samples, each of size $n$, are taken from the same continuous distribution. $U_r^n$, the number of exceedances, is the number of values in the second sample exceeding the $r$th smallest in the first sample, $r$ fixed. Taking advantage of certain symmetries, it is possible to give the complete distribution function, Pr $(U_r^n \leq x)$, in terms of $r$ up to $[n/2]$ and $r - 1 \leq x \leq n - r - 1$. Table 1 gives these to not less than 4D and 3S for $n = 2(1)15(5)20$. [WAYNE UNIVERSITY Technical Report No. 6, by the same author, gives values for $n = 2(1)20(5)50$.] Table 2 indicates, for the case $n = 5$, how to complete the array of probabilities from the portion given in Table 1.

LEO KATZ

Michigan State University
East Lansing, Michigan

100[K].—C. K. TSAO, "An extension of MASSEY's distribution of the maximum deviation between two-sample cumulative step functions," *Ann. Math. Stat.*, v. 25, 1955, p. 587–592.

Let $S_n(x)$ and $S_m'(x)$ be the cumulative distributions observed in two random samples, of sizes $n$ and $m$ respectively, from the same continuous population. Denote by $x_r$ and $y_r$ the $r$th smallest in the respective samples, and for $r \leq \min(m, n)$ let

$$d_r = \max_{x \leq x_r} |S_n(x) - S_m'(x)|$$

and

$$d_r' = \max_{x \leq \max(y_r, x_r)} |S_n(x) - S_m'(x)|.$$

Under the above assumption the distributions of $d_r$ and $d_r'$ are independent of the distribution of the population. Table I gives $P(d_r \leq c/m)$ to 5D for $n = m = 3(1)10, 15, 20, 30, 40$; $c = 1(1)12$; $r = 2(1)10$. Table II gives $P(d_n' \leq c/m)$ to 5D for the same range of arguments.

These tables may be used in testing the hypothesis that two populations have identical distribution functions.

FRANK MASSEY

University of Oregon
Eugene, Ore.

**101[K].**—YOSHIHIKO HIRAGO, HIDENORI MORIMURA, & HISAO WATANABE, "Tables for three-sample test," Inst. Stat. Math., *Annals*, v. 5, 1954, p. 97–102.

The authors provide tables for a generalization, to three samples, of a two-sample test for randomness originated by WALD and WOLFOWITZ [1] and studied in greater detail by MOOD [2].

Let $x_1, x_2, \cdots, x_{n_1}$, $y_1, y_2, \cdots, y_{n_2}$, and $z_1, z_2, \cdots, z_{n_3}$, $n_1 \leq n_2 \leq n_3$, be three random samples drawn from the respective populations having continuous cdf's $F(x)$, $G(y)$, $H(z)$. To test whether the $n = n_1 + n_2 + n_3$ observations form a single random sequence from the same population, i.e., whether $F \equiv G \equiv H$, the values are arranged in a single ascending order and the runs of $x$, $y$, $z$ counted, regardless of length. Let $r_1, r_2, r_3$ be the number of runs of $x$'s, $y$'s, $z$'s, respectively and put $r = r_1 + r_2 + r_3$. For given values of $n_1$, $n_2$, $n_3$ the critical value of $r$ at the $\alpha$-level of significance is defined as the largest integer, $r_0$, for which $\text{Prob}[r \leq r_0] \leq \alpha$. Table 1, for $\alpha = .05$, is for all $n_i$ for which $1 \leq n_1 \leq n_2 \leq n_3 \leq 10$. It was obtained from the exact distribution of $r$ discussed by Mood in the above paper. Table 2, for $\alpha = .05$, is for $10 \leq n_1 \leq n_2 \leq n_3 \leq 30$, the $n_i$ selected from the set 10(2)20(5)30, but excluding $n_3 = n_1$. Table 3 gives the $\alpha = .01$ and .05 critical values for all $n_i$ for which $11 \leq n_1 = n_2 = n_3 \leq 30$. Tables 2 and 3 were obtained by using the normal approximation given by Mood with a correction based on his formulas for the first two moments of $r_i$.

J. LIEBLEIN

Statistical Engineering Laboratory
National Bureau of Standards
Washington, D. C.

1. A. WALD & J. WOLFOWITZ, "On a test whether two samples are from the same population," *Ann. Math. Stat.*, v. 11, 1940, p. 147–162.
2. A. M. MOOD, "The distribution theory of runs," *Ann. Math. Stat.*, v. 11, 1940, p. 367–392.

**102[K].**—W. J. DIXON, "Power under normality of several nonparametric tests," *Ann. Math. Stat.*, v. 25, 1954, p. 610–614.

Consider a sample of size $N_1$ from a normal population with mean $\mu_1$ and variance $\sigma^2$ and an independent sample of size $N_2$ from a normal population with mean $\mu_2$ and variance $\sigma^2$. This paper presents tabulations of the power and power efficiency [1] of the rank-sum, maximum deviation, median, and total number of runs tests for the null hypothesis $\mu_1 = \mu_2$ with significance level $\alpha$ and alternative hypotheses $\delta = |\mu_1 - \mu_2|/\sigma$. Table I contains power function values to 3D and power efficiencies to 2S for the cases $\alpha = 1/10, 1/35, 1/126$, $N_1 = N_2 = 3(1)5$, $\delta = 0(.25)2(.5)5$ for all tests; also for the cases $\alpha = 2/126, 4/126$, $N_1 = N_2 = 5$, $\delta = 0(.5)4.5$ for the rank sum test; for the cases $\alpha = 10/126$, $N_1 = N_2 = 5$, $\delta = .5(.5)4.5$ for the maximum absolute deviation test; and for the cases $\alpha = 26/126$, $N_1 = N_2 = 5$, $\delta = .5(.5)3$ for the median test. Table II contains power function values to 3D and power efficiencies to 2S for all tests except

the total number of runs test for a randomized significance level of $\alpha = .025$, $N_1 = N_2 = 5$, $\delta = .5(.5)4.5$. Table III gives limiting power efficiencies as $\delta \to 0$ to 3S for the rank sum test for the cases: $\alpha = 1/3$ and $(N_1, N_2) = (2, 2)$; $\alpha = 1/5$ and $(N_1, N_2) = (2, 3)$; $\alpha = 2/15, 4/15$ and $(N_1, N_2) = (2, 4)$; $\alpha = 1/10$, $1/5$ and $(N_1, N_2) = (3, 3)$; $\alpha = 1/70$ and $(N_1, N_2) = (4, 4)$; $\alpha = 1/126$, $2/126$, $4/126$ and $(N_1, N_2) = (5, 5)$; $0 < \alpha < 1$ and $(N_1, N_2) = (\infty, \infty)$.

J. E. WALSH

Military Operations Research Division
Lockheed Aircraft Corporation
Burbank, California

1. W. J. DIXON, "Power functions of the sign test and power efficiency for normal alternatives," *Ann. Math. Stat.*, v. 24, 1953, p. 467–473.

103[K].—E. S. PAGE, "Control charts for the mean of a normal population," Roy. Stat. Soc., *Jn.* s. B., v. 16, 1954, p. 131–135.

Consider the well known statistical quality control procedure in which the method is to examine a fixed number of items at regular intervals, to measure a certain dimension, $x$, and to plot the sample mean, $\bar{x}$, on a control chart. If the point falls outside the control limits drawn at $\mu \pm \frac{B\sigma}{\sqrt{N}}$ (where $\mu$ is the required process mean, and $\sigma$ is the process standard deviation), the process mean is said to be "out of control" and some appropriate action is taken. However, it is the premise of this paper that unnecessary restriction exists when the width of these control limits is fixed, and the author proceeds to present a method by which $B$ and $N$ are chosen to obtain a "best" scheme.

For an estimated process mean, $m$ (perhaps different from $\mu$), and a known constant standard deviation, $\sigma$, it is desired to detect any change in the mean. Consider only the case in which we are interested in a single control limit at $\mu + B\sigma/\sqrt{N}$ and a constant process mean, $m$. Let $P(m)$ denote the probability that a given sample yields a point outside the control limit. Define the "average run length," $L$, as the average number of articles inspected between two successive occasions when action is taken. The average run length for a constant process mean, $m$, is $L = N/P(m)$.

The average number of articles produced between two successive occasions when action is taken is $L/f$ where $f$ is the fraction of output that is inspected. It is easily seen that the average run length is a measure of the amount of scrap or defective articles produced when quality is bad, and is also a measure of the amount of acceptable articles produced between interferences caused by process inspection when the quality is satisfactory. If $\mu$ is a satisfactory value for the mean, $m$, then for $m = \mu$ there is no need for any corrective action to be taken. It is desirable that the production should only rarely be suspended. Define $L_0$ as the average run length when $m = \mu$. Let $k$ (a positive number) be such that a shift in the mean of $m \geq \mu + k\sigma$ should be detected as soon as possible after it has occurred. Define $L_1$ as the average run length when $m = \mu + k\sigma$. A "best" inspection scheme might be defined as one that minimizes $L_1$ for some given large value of $L_0$, or, alternately, as one which maximizes $L_0$ for some given small value of $L_1$.

If only deviations from the mean in one direction are considered, the control limit is $\mu + \frac{B\sigma}{\sqrt{N}}$ and $L_0 = \frac{N}{\Phi(B)}$, $L_1 = N/\Phi(B - k\sqrt{N})$, where

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_x^\infty \exp(-x^2/2)dx.$$

In accord with the above principles for the one-sided scheme, Tables 1a–c give $N$ and $B$, the latter to 2D, which minimize $L_1$ for which $L_0 = 2{,}000$, 5,000(5,000)20,000, 40,000, 60,000 is specified for a given $k = .2(.1)1.8$. Also Tables 2a–c give $N$ to 2 or 3S and $B$ to 2D which maximize $L_0$ for which $L_1 = 10(5)25(25)100$ is specified for a given $k = .1(.1)1$.

G. W. McELRATH

University of Minnesota
Minneapolis, Minn.

**104[K].**—T. E. STERNE, "Some remarks on confidence or fiducial limits," *Biometrika*, v. 41, 1954, p. 275–278.

Define $P_{n,r}(\pi) = \binom{n}{r}\pi^r(1-\pi)^{n-r}$ and $P(\pi, n, s) = \sum_r P_{n,r}(\pi)$ for all $r$ for which $P_{n,r}(\pi) \leq P_{n,s}(\pi)$. From these it is possible to find confidence limits corresponding to any $n$, $s$, and any desired value $\epsilon$, where $n$ is the number of trials and $s$ the number of successes. It is sufficient to consider the shortest interval of $\pi$ that contains all values of $\pi$ for which $P(\pi, n, s) \geq \epsilon$ for a given $n$ and $s$. The lower limit is the smallest $\pi$ satisfying the inequality and the upper limit is the largest $\pi$. Tables for confidence limits for $\pi$ to 2 and 3D with confidence coefficients 0.5 and 0.9 are given for all possible values of $s$ when $n = 1(1)10$.

J. R. VATNSDAL

State College of Washington
Pullman, Washington

**105[K].**—H. A. DAVID, H. O. HARTLEY, & E. S. PEARSON, "The distribution of the ratio, in a single normal sample, of range to standard deviation," *Biometrika*, v. 41, 1954, p. 482–493.

Percentage points of the distribution of $w/s$, where $w$ is the range and $s$ the sample standard deviation of a sample of $n$ from a normal population, are obtained partly by the use of an exact distribution applicable in the case of small samples, and partly by fitting a curve of the PEARSON system to the first four moments for selected values of $n$ and interpolating between these results. The final table gives upper $\alpha$ percentage points for $n = 3(1)20(10)60(20)100$, 150, 200, 500, 1000 and $\alpha = 0.10$, 0.05, 0.025, 0.01, 0.005 to 3D for $n \leq 9$ and 2D for $n > 9$, and lower percentage points for the same values of $\alpha$ and $n \geq 10$. The use of this ratio as a test of homogeneity in both large and small samples is considered, and it is compared to the ratio of the mean deviation to the sample standard deviation and the sample fourth standard moment as a test of normality.

C. A. BENNETT

General Electric Company
Hanford Atomic Products Operation
Richland, Washington

**106[K].**—MARTIN WEIBULL, "The distribution of *t*- and *F*-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means," *Skandinavisk Akuarietidskrift*, v. 36, 1953, supplement to haft 1–2.

This long paper is principally concerned with the application of the standard statistics used in exact significant tests in samples from normal to samples from stratified universes in which the distributions within each stratum are normal with variances constant in all strata but with means different in different strata. It begins with a discussion of non-central $\chi^2$, *t*- and *F*-statistics and includes an illustrative table of the non-central *F* distribution in which both the $\chi^2$'s appearing in the numerator and denominator are non-central. 1% and 5% points of this *F* are given to 2D for the degrees of freedom for the numerator, $b_1 = 1, 2, 4, 8$, the degrees of freedom for the denominator, $b_2 = 1, 2, 4, 8, 16, 32, \infty$, but only for $b_1 < b_2$, and the non-centrality parameters $\beta_1$ and $\beta_2 = 0, \frac{1}{2}, 1$.

G. J. LIEBERMAN

Stanford University
Stanford, Calif.

**107[K].**—A. J. DUNCAN, "The use of ranges in comparing variabilities," *Industrial Quality Control*, v. 11, no. 5, 1955, p. 18–22.

The author tabulates the quantities $d_2^*$ and $\nu$ such that $E(\bar{R}/d_2^*)^2 = \sigma^2$ and the mean and variance of $\sqrt{\nu}\bar{R}/d_2^*\sigma$ are those of a $\chi^2$-distribution with $\nu$ degrees of freedom, where $\bar{R}$ is the mean of the ranges of $k$ samples of $n$ from a normal distribution with variance $\sigma^2$. $d_2^*$ is given to 2D and $\nu$ to 1D for $k = 1(1)20, 25, 30, 50$ and $n = 2(1)7$. This is an extension of Table I of DAVID [1] where $c = d_2^*$ is given to 2D and $\nu$ to 1D for $k = 1(1)5, 10$ and $n = 2(1)10$, which in turn is an extension, with minor corrections, of the original table given by PATNAIK [2]. The methods used for the extension are not indicated.

C. A. BENNETT

Hanford Atomic Products Operation
General Electric Company
Richland, Washington

1. H. A. DAVID, "Further applications of range to the analysis of variance," *Biometrika*, v. 38, 1951, p. 393–407.
2. P. B. PATNAIK, "The use of mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

**108[K].**—T. LEWIS, "99.9 and 0.1% points of the $\chi^2$ distribution," *Biometrika*, v. 40, 1953, p. 421–426.

Points of the $\chi^2$ distribution have been computed correct to 8D for the lower 0.1% points for degrees of freedom $n = 1(1)30(10)100, 120$, and for the upper 0.1% points for $n = 40(10)100, 120$.

Using $m = \frac{1}{2}n$, $v = \frac{1}{2}\chi^2$, and $P = \dfrac{1}{\Gamma(m)}\displaystyle\int_v^\infty t^{m-1}e^{-t}dt$, so that $2v$ is the 100P% point, the author had $f(v) = 0$ where for $n$ even,

$$f(v) = \left\{1 + v + \frac{v^2}{2!} + \cdots + \frac{v^{m-1}}{(m-1)!}\right\} - \mathrm{Pe}^v$$

and for $n$ odd,

$$f(v) = \left\{\sqrt{\frac{2}{\pi}}\, e^{v} \int_{\sqrt{2v}}^{\infty} e^{-\frac{1}{2}t^2} dt + \frac{v^{\frac{1}{2}}}{(1/2)!} + \frac{v^{\frac{3}{2}}}{(3/2)!} + \cdots + \frac{v^{m-1}}{(m-1)!}\right\} - \mathrm{P}e^{v}.$$

Then each value of $v$ was obtained by computing $f(v)$ approximately for an approximate value $v_0$ of $v$ and then improving by iteration. The first $v_0$ was obtained by a formula due to WILSON and HILFERTY [1],

$$\sqrt[3]{\frac{v}{m}} \sim 1 - \frac{1}{9m} + x\sqrt{\frac{1}{9m}},$$

the $x$ being the normal deviate corresponding to the 100P% value of $v$.

Tables of these computations are exhibited.

T. A. BICKERSTAFF

University of Mississippi
University, Miss.

1. E. B. WILSON & M. M. HILFERTY, "The distribution of chi-square," Nat. Acad. Sci., *Proc.*, v. 17, 1931, p. 684–688.

**109[K].**—S. H. ABDEL-ATY, "Approximate formulae for the percentage points of the non-central $\chi^2$ distribution," *Biometrika*, v. 41, 1954, p. 538–540.

The main point of this note is to derive the two approximation formulas of the title and compare them numerically with the exact values. The cube root transformation of WILSON and HILFERTY turns out to be the appropriate one for transforming the non-central $\chi^2$ variate, $\chi'^2$, to an approximately normal variate $y$. The CORNISH-FISHER expansion [1] is then used to obtain a formula for the $100\alpha$-percentage point of $Y$ (the variate $y$ after standardization) in ascending powers of $r^{-\frac{1}{2}}$, where $r = f + \lambda$ and $f$ is the number of degrees of freedom and $\lambda$ is the parameter. This formula is written out to the term in $r^{-\frac{1}{2}}$, with the coefficients expressed in terms of quantities whose values are given for $\alpha = .5(.1).1, .05, .01, .005, .001$. The numerical comparison with a few exact values indicates that even for small degrees of freedom and moderate values of $\lambda$ the expansion agrees very well for both the upper and lower 5-percent points. The probability integral approximation for $Y$ is obtained by straightforward application of the EDGEWORTH expansion. Numerical comparison with exact values for a few values of $f$ and $\lambda$ show agreement to 3D or better.

JULIUS LIEBLEIN

Statistical Engineering Laboratory
National Bureau of Standards
Washington 25, D. C.

1. E. A. CORNISH & R. A. FISHER, "Moments and cumulants in the specification of distributions," *Extrait de la Revue de l'Institut International de Statistique*, v. 4, 1937, p. 1–14.

**110[K].**—A. M. FREUDENTHAL & E. J. GUMBEL, "Minimum life in fatigue," Amer. Stat. Assn., *Jn.*, v. 49, 1954, p. 575–597.

The authors fit fatigue data by the "survivorship function"

$$l(N)_S = \exp\left[-\left(\frac{N - N_{0,S}}{V_S - N_{0,S}}\right)^{\alpha_S}\right].$$

In this formula, $l(N)_S$ is the probability of surviving $N$ cycles under stress $S$; $V_S$ and $N_{0,S}$ are location parameters having dimensions of $N$ and the relation $V_S > N_{0,S} \geq 0$; $N_{0,S}$ stands for the minimum life measured in cycles to failure at the stress level $S$; $1/\alpha_S$ is a dimensionless scale parameter. The problem of estimating the three unknown parameters $\alpha_S$, $N_{0,S}$, and $V_S$ from a set of fatigue data taken at a series of stress levels $S$ is a formidable one. The authors use the method of moments to carry out the estimation. This procedure is facilitated by the authors' Table 1. This table gives for $1/\alpha_S = 0(.01)1(.2)2(1)5$ the values to 4D of the standardized distance from the "characteristic number," $\sqrt{S}$ to the minimum life, of the standardized distance from $\sqrt{S}$ to the mean, and of the third central moment in standard units. The method is applied to published data of RAVILLY on the fatigue strength under tension of aluminum and nickel wire. The question of the sampling errors of estimate obtained by the authors' procedure is left open.

BENJAMIN EPSTEIN

Wayne University
Detroit, Michigan

111[K].—MAX HALPERIN, SAMUEL W. GREENHOUSE, JEROME CORNFIELD, & JULIA ZALOKAR, "Tables of percentage points for the Studentized maximum absolute deviate in normal samples," Am. Stat. Assn., *Jn.*, v. 50, 1955, p. 185–195.

Let $x_i$, $i = 1$ to $k$, be independent normally distributed variates each with mean $\mu$ and variance $\sigma^2$. The Studentized maximum absolute deviation is defined by $d = \max_{i=1,2,\cdots,k} \frac{|x_i - \bar{x}|}{s}$, where $ms^2/\sigma^2$ is distributed as $\chi^2$ with $m$ degrees of freedom and independent of $x_i$. Tables of the upper and lower limit for the upper 5% and 1% points of $d$ are given to 3S for $k = 3(1)10(5)20(10)40$, 60 and $m = 3(1)10(5)20(10)40$, 60, 120, $\infty$. Examples illustrate the use of the tables.

L. A. AROIAN

Hughes Aircraft Co.
Culver City, Calif.

112[K].—J. H. CADWELL, "The probability integral of range for samples from a symmetrical unimodal population," *Ann. Math. Stat.*, v. 25, 1954, p. 803–806.

An asymptotic expression is given for the probability integral of range for samples from a symmetrical unimodal population. Its accuracy is investigated for the case of a normal parent population for sample sizes of 20 to 100. Over this range errors are small and by use of a given table of corrections the probability integral can be found with a maximum error of 0.0001. The .1, .5, 1, 5, 10, 25, 50, 75, 90, 95, 99, 99.5, 99.9 percentage points are given to 3S, for sample sizes of 20(20)100 of the range $w$ from a normal population with unit standard deviation and also the mean value of $w$ for the same sample sizes.

L. A. AROIAN

Hughes Aircraft Company
Culver City, California

**113[K].**—S. R. BROADBENT, "The quotient of a rectangular or triangular and a general variate," *Biometrika*, v. 41, 1954, p. 330–337.

The paper discusses the distribution of the quotient of the value of a random variable from a rectangular or a triangular distribution and the value of a random variable from an independent distribution. The denominator distribution is restricted to one for which a frequency function exists almost everywhere and for which, if the cumulative distribution function is $F(x) = z$, the inverse function, $G(z) = x$, is defined, non-zero, and finite for almost all $z$.

The tables of the paper are for the cases in which the denominator distribution is a "normal" distribution. Table 1 is entitled "Percentage points of the quotient of a rectangular and an independent normal variate. If $y$ is rectangular with range $(1 - \alpha, 1 + \alpha)$ and $x$ is normal with mean 1 and variance $\beta^2$, the table gives percentage points of $(y/x)$." The values given are solutions for $Q$ to 3D of [1]

$$Hh_0\left(-\frac{1}{\beta}\right) + \frac{\beta Q}{2\alpha\sqrt{2\pi}}\left\{Hh_1\left(\frac{1-\alpha}{\beta Q} - \frac{1}{\beta}\right) - Hh_1\left(\frac{1+\alpha}{\beta Q} - \frac{1}{\beta}\right)\right\} = k,$$

$k = .01, .05, .95, .99$; $\alpha = 0(.02).10$; $\beta = 0(.01).05$. Table 2 is entitled "Percentage points of the quotient of a triangular and an independent normal variate. If $y$ is triangular with range $(1 - 2\alpha, 1 + 2\alpha)$, and $x$ is normal with mean 1 and variance $\beta^2$, the table gives percentage points of $(y/x)$." The values given are solutions for $Q$ to 3D of

$$Hh_0\left(-\frac{1}{\beta}\right) + \frac{\beta^2 d^2}{(2\alpha)^2\sqrt{2\pi}}\left\{Hh_2\left(\frac{1-2\alpha}{\beta Q} - \frac{1}{\beta}\right) - 2Hh_2\left(\frac{1}{\beta Q} - \frac{1}{\beta}\right) + Hh_2\left(\frac{1+2\alpha}{\beta Q} - \frac{1}{\beta}\right)\right\} = k,$$

$k = .01, .05, .95, .99$; $\alpha = .00(.01).05$; $\beta = .00(.01).05$.

In all cases tabled, $Hh_0(-1/\beta)$ is negligible and could be omitted from the above formulas without affecting the solution.

K. J. ARNOLD

Michigan State University
East Lansing, Mich.

1. For the definition of $Hh_n(x)$ see *BAAS Math. Tables*, v. I, London, 1941, p. x.

**114[K].**—S. B. CHOWDHURY, "The most powerful unbiased critical regions and the shortest unbiased confidence intervals associated with the distribution of the classical $D^2$-statistic," *Sankhyā*, v. 14, 1954, p. 71–80.

This continuation of an earlier paper by P. K. BOSE [1] includes two short tables for determining the shortest unbiased upper and lower 95% confidence intervals for a parameter $\Delta^2$ used in testing the equality of means of two $p$-variate normal populations. Let the (known) variance matrices of the two populations be $A = (\alpha_{ij})$, $A' = (\alpha'_{ij})$, and their unknown vector means be $\{\alpha_i\}$, $\{\alpha_i'\}$. In terms of these quantities,

$$\Delta^2 = p^{-1}\sum\sum\beta^{ij}(\alpha_i - \alpha_i')(\alpha_j - \alpha_j'),$$

where $\beta^{ij}$ are the elements of the inverse matrix $(\beta_{ij})^{-1}$, $\beta_{ij} = (n'\alpha_{ij} + n\alpha'_{ij})/(n+n')$. To test the equality of means a random sample is drawn from each population, of, say, respective sizes $n$, $n'$, and sample means $\{a_i\}$, $\{a_i'\}$. One then computes a sample statistic $D_1^2$ (to which the "classical $D^2$" statistic introduced by MAHALANOBIS is related by $D^2 = D_1^2 - 2/\bar{n}$), corresponding to $\Delta^2$, by means of $D_1^2 = p^{-1}\sum\sum\beta^{ij}(a_i - a_i')(a_j - a_j')$, and from this, the quantities $l^2 = kD_1^2/2$, $L = (\bar{n})^{\frac{1}{2}}l$, where $\bar{n}$ is the harmonic mean of $n$, $n'$.

Table 1 gives, for $p = 1(1)10$ and for 14 values of $L$ ranging from $L = 2$ to $L = 400$, the lower 95% confidence limit $\lambda$ for the parameter $\lambda$ related to $\Delta^2$ in the same way as $l$ is related to $D_1^2$, for all alternatives $\lambda > \lambda_0$. Here $\lambda_0$ represents the hypothesis to be tested, that is, $\lambda_0$ represents the bound for $\lambda$ which would assure that $\Delta^2$ is sufficiently small to make the means equal within desirable tolerances. The values of $\lambda = \lambda(L)$ are the solutions, for the given values of $L$, of

$$\int_0^L \psi(x, \lambda(L))dx = 1 - \alpha,$$

where $\alpha = .95$, and

$$\psi(L, \lambda) = L^{p/2}\lambda^{-(p-2)/2}e^{-(L^2+\lambda^2)/2}I_{(p-2)/2}(L\lambda)$$

(where $I$ is the imaginary Bessel function) is the probability density function of $L$.

Table 2 gives, for the same values of $p$ and $L$, the upper 95% confidence limit $\bar{\lambda}$, for all alternatives $\lambda < \lambda_0$, in precisely analogous manner to Table 1, with the limits of integration 0 to $L$ in the equation for $\bar{\lambda}(L)$ being replaced by $L$ to $\infty$.

JULIUS LIEBLEIN

Statistical Engineering Laboratory
National Bureau of Standards
Washington 25, D. C.

1. P. K. BOSE, "On recursion formulae, tables, and Bessel function populations, associated with the distribution of classical $D^2$-statistic," *Sankhyā*, v. 8, 1947, p. 235–248.

**115[K].**—D. R. COX, "The mean and coefficient of variation of range in small samples from non-normal populations," *Biometrika*, v. 41, 1954, p. 469–481.

The author is interested in the mean range and the coefficient of variation of the range in small samples of 2, 3, 4, and 5, from non-normal populations. Different types of populations covering a wide range of values of $\beta_1$ and $\beta_2$, the usual measures of skewness and kurtosis, are considered: symmetrical and unsymmetrical mixed normal distributions, the normal distribution, the rectangular distribution, exponential type distributions, the PEARSON system, and SHONE's [1] numerical results for 5 populations of discrete values. Based on these results he provides a table for the normalized mean range and the coefficient of variation of the range to 3D for sample sizes of 2, 3, 4 and 5, for $\beta_2 = 1(.2)2(.5)5(1)9$; $\beta_1$ is not a determining factor. Comparisons are made between the distribution function of the range and the ratio of two ranges from the exponential $e^{-x}$ and the unit normal population. The theory is applied to the point estimation of dispersion by use of the range, the control chart for the range, the comparison of several

ranges by use of the range ratio test and the use of the range in the $t$-test for non-normal as well as normal distributions. Many short tables give numerical illustrations. The author concludes that if $\beta_2$ is known, it is possible to make a rough correction for non-normality in methods that use ranges of small samples, and that on the whole these methods are less affected by non-normality than corresponding methods using variances of small samples.

L. A. AROIAN

Hughes Aircraft Company
Culver City, Calif.

1. K. J. SHONE, "Relations between the standard deviation and the distribution of range in non-normal populations," Roy. Stat. Soc., *Jn.*, s.B., v. 11, 1949, p. 85–88.

**116[K].**—H. A. DAVID, "The distribution of the range in certain non-normal populations," *Biometrika*, v. 41, 1954, p. 463–468.

Explicit expressions for the probability integral $P(w|n)$ and the expectation $E(w|n)$ of the range $w$ of a sample of $n$ are obtained for several non-normal distributions with varying degrees of skewness and/or kurtosis. Values of $Q(w|n) = 1 - P(w|n)$ for $w = k(n, \alpha)\sigma_x$, where $k(n, \alpha)$ is the $100\alpha\%$ point of $w/\sigma$ for the normal distribution, are given to 3D in Table 1 for $\alpha = .01, .05$, $n = 2(2)12$, and various degrees of skewness in the populations considered. Part of these values were obtained from the explicit expressions and part by direct quadrature using related tables. Table 2 gives the ratio $E(w|n)/d_n\sigma_x$ to 3D, where $d_n$ is the expected range, in standard units, of a sample of $n$ from the normal distribution, for $n = 2(2)12$ and those distributions for which $E(w|n)$ was obtained explicitly. Table 2 indicates that appreciable non-normality has little effect on the use of the range to estimate the standard deviation, but Table 1 further substantiates the difficulty in distinguishing between changes in variability and departures from normality.

C. A. BENNETT

General Electric Company
Hanford Atomic Products Operation
Richland, Michigan

**117[K, L].**—H. J. GAWLIK, *A table of a function related to the error function*, Armament Research and Development Establishment, BR. Memo. B4/1/55, Fort Halstead, Sevenoaks, Kent. 3 typescript pages mimeographed, 10 leaves of tables (photo-offset) stapled, 33 cm. Deposited in the UMT FILE.

The table lists $F(x)$ to 10D for $x = 0(.01)10.00$, where

$$F(x) = e^{-x^2/2} \int_0^x e^{t^2/2} dt.$$

There is a brief discussion of the computational techniques used and the author's estimate that all digits are accurate except for possible round off inaccuracy in the last digit. There is no discussion of use of the table or interpolation. The calculation was on the AMOS machine.

C. B. T.

**118[L].**—OWEN R. MOCK, *Table of Bessel functions at multiples of $\pi/2$.* 8 leaves (mimeographed both sides), deposited in UMT FILE.

A table of $J_n(x)$ for $n = 0(1)59$ and $x = k\pi/2$ to 20D for all non-negative integral values of $k$ giving significant values at this precision and not exceeding 25. There is no introduction or description. The tables are believed to have been computed on a Model 701 International Business Machine, and the author believes that all digits are correct.

C. B. T.

**119[L].**—ALBERT D. WHEELON & JOHN T. ROBACKER, *A Table of Integrals involving Bessel Functions*, The Ramo-Wooldridge Corporation, Los Angeles, California, 1954, 91 p., 28 cm. Multilithed from manuscript and typescript. Deposited in UMT FILE.

The authors note that the preparation of this table of 400 integrals was halted when they learned of the BATEMAN Manuscript Project which was doing similar work. They hope the tables contain the most frequently encountered integrals. This set is much less comprehensive than those in the Bateman Manuscript Project, "Tables of Integral Transforms," v. 2, New York, 1954, p. 5–177. There are, however, a few formulas in the present work which are not found in "Tables of Integral Transforms."

The authors' careful indication of sources of their formulas is valuable.

Sometimes the statement of the formula is careless. In particular, the reviewer noted the following:

p. 40, 11.306. The important condition $b \leq a$ is not stressed.
p. 41, 11.311. The interpretation of the formula for $a \geq 1$ might be noted.
p. 46, 11.201. Many similar integrals published by W. N. BAILEY, "Some infinite integrals involving Bessel functions," London Math. Soc., *Proc.*, v. 40, 1936, p. 37–48, are not listed. However, this particular integral seems new.
p. 15, 1.207. The integral on the left is divergent. The authors probably intend to have the principal value used.

The indexing scheme is clearly described, and integrals are easily found in the list.

P. HENRICI

National Bureau of Standards
Washington, D. C.

**120[M, S].**—A. J. W. DUIJVESTIJN & J. BERGHUIS, *The computation and the expansion of some triple integrals originating from the theory of cosmic rays.* R 261, *Math. Centrum*, Amsterdam. Seventeen leaves mimeographed from typescript deposited in the UMT FILE.

This work contains tables of $K$ and $J(r)$ for $a = 0.005, 0.01, 0.05, 0.1, 0.5, 1$ and $J^*(B)$ for $a = 0.1$ and 1. Two cases are considered: (1) $\alpha = 0.0245/r^3$, $\beta = 0.0245/r_1^3$, and $\gamma = 0.0245/r_2^3$; (2) $\ln \alpha = \ln 0.454 - \ln r + \ln (1 + 4r) - 4r^{2/3}$, $\ln \beta = \ln 0.454 - \ln r_1 + \ln (1 + 4r_1) - 4r_1^{2/3}$, $\ln \gamma = \ln 0.454 - \ln r_1 + \ln (1 + 4r_2) - 4r_2^{2/3}$.

Here $r_{1,2} = (a^2 + r^2 \pm 2ar\cos\theta)^{1/2}$, and

$$I(r, B) = r\int_{\theta=0}^{\pi/2} (1 - e^{-B\alpha})(1 - e^{-B\beta})(1 - e^{-B\gamma})d\theta$$

$$J^*(B) = B^{-5/2}\int_{r=0}^{\infty} I(r, B)dr$$

$$J(r) = \int_{B=0}^{\infty} B^{-5/2}I(r, B)dB$$

$$K = \int_{r=0}^{\infty} J(r)dr = \int_{B=0}^{\infty} J^*(B)dB.$$

Behavior of $J^*(B)$ for small and large positive values of $B$ is discussed, and the method of computation is described.

The integrals are of interest in studies of cosmic rays.

C. B. T.

## TABLE ERRATA

In this issue references have been made to errata in Review **80**, Review **81**, and Review **89**.

In *MTAC*, 9, July 1955, reference was made to errata in Review **55** and Review **59**.

**245.**—E. JAHNKE & F. EMDE, *Tables of Functions*. Dover edition, 1945.

On p. 211, section 9, chapter VIII, *the left side of the last formula should read* $-\Omega_p(z) + N_p(z)\cdots$ *instead of* $\Omega_p(z) + N_p(z)\cdots$.

HENRY E. FETTIS

931 Five Oaks Avenue
Dayton 6, Ohio

**246.**—ROBERT E. GREENWOOD, "Coupon Collector's Test for Random Digits," *MTAC*, 9, 1955, p. 3.

Entry for $n = 13$ *should read* .00800 8315 2 *instead of* .0080 9315 2.

ROBERT E. GREENWOOD

The University of Texas
Austin, Texas

## NOTES

### R. C. Archibald

Professor RAYMOND C. ARCHIBALD, who founded *Mathematical Tables and Other Aids to Computation* and who was always one of its principal contributors and the most valued adviser to the Editorial Committee, died on July 26, 1955 at Sackville, New Brunswick, Canada.

Professor Archibald was the Chairman of the Editorial Committee from the first issue of the journal in 1943 through 1949.

A short history of some of Professor Archibald's professional and personal accomplishments will appear in *MTAC* shortly.

There can be no adequate statement of the loss felt by the Editorial Committee and other people interested in mathematical tables, computation, and related topics on learning of Professor Archibald's death.

C. B. T.

## Postdoctoral Research Associates of the National Bureau of Standards

The National Academy of Sciences, the National Research Council, and the National Bureau of Standards announced a plan for postdoctoral research associates at the National Bureau of Standards in October 1954. The program is intended to provide advanced training, through Bureau sponsored research and through close association with outstanding members of the staff of the Bureau, to a few carefully selected young scientists.

This program is directed by Dr. WALLACE R. BRODE, Mr. JOSEPH HILSENRATH, and Dr. DAVID E. MANN of the National Bureau of Standards.

Three appointments for the current year are in numerical analysis or closely related fields.

Dr. RICHARD J. PROSEN plans to apply the computer SEAC to an investigation of crystal structure, with particular emphasis on problems of atomic arrangement. He hopes to answer the question, "How much and how well can an electronic computer be used in such problems?" (Mr. HOWARD R. MCMURDIE, NBS advisor.)

Dr. MORRIS KRAUSS plans a theoretical quantum mechanical study of the relationship between electron energy levels and molecular structure. Such calculations may help explain certain basic problems that are important in other fields such as mass spectrometry. SEAC will be the primary tool used in the study. Dr. KRAUSS will attempt to answer two questions: "How do you go about solving such a highly complex problem?" and "Of what value are known methods?"

## National Simulation Conference

The Dallas-Fort Worth Chapter of the Institute of Radio Engineers Professional Group on Electronic Computers (PGEC) will sponsor a National Simulation Conference in Dallas, Texas, on 19–21 January 1956.

The Conference will be devoted to simulation and associated computing techniques, and will include topics in (a) general simulation (mathematical, physical, logistic, etc.); (b) advances in computer design, techniques, and applications; and (c) methods of determining and improving the accuracy of analog solutions.

While the I.R.E. does not sponsor meetings requiring military security clearance, an attempt will be made to arrange such a session on an "associated" basis if interest warrants.

It is planned to publish "proceedings" of the Conference.

Further information regarding the Conference can be obtained from

Mr. J. R. Forester
2104 Huntington
Arlington, Texas.

## The Second Regional Graduate Summer Session in Statistics held at the University of Florida in June and July 1955

At the request of the Southern Regional Education Board's Advisory Commission on Statistics, the University of Florida, Virginia Polytechnic Institute and North Carolina State College agreed to initiate a continuing program of graduate summer sessions in statistics to be held at each of the three institutions in rotation, beginning in the summer of 1954. At the first session held at Virginia Polytechnic Institute eighty-nine students were enrolled, and at the second session ninety-six students were enrolled. The third session will be held at North Carolina State College in Raleigh in the summer of 1956.

It is the purpose of this program to serve: (1) research and professional workers who want intensive instruction in basic statistical concepts and modern statistical methodology; (2) teachers of elementary statistical courses who want formal training in modern statistics; (3) graduate students in other fields who desire supporting work in statistics; (4) prospective candidates for graduate degrees in statistics, and (5) professional statisticians who wish to keep informed about advanced specialized theory and methods.

Each of the summer sessions will last six weeks and each course will carry approximately three semester hours of graduate credit.

## Sponsorship by the Office of Naval Research, the National Bureau of Standards, the National Science Foundation, and various other organizations

Much of the editorial and refereeing work for this and for earlier volumes of *Mathematical Tables and Other Aids to Computation* has been done under the sponsorship of the Office of Naval Research and the National Bureau of Standards as part of their program for the development of numerical analysis and research connected with numerical analysis. Considerable help has been given to the Editorial Committee by various committees sponsored by the National Science Foundation and several learned societies in connection with mathematical and statistical tables. Finally, the members of the Editorial Committee who are members of University faculties and the referees of papers submitted to the journal have carried out their work for *Mathematical Tables and Other Aids to Computation* as part of their University research assignment. The contributions of all these organizations to the publication of this journal are gratefully acknowledged.

## Index

In this issue the Editorial Committee has decided to try to set a format for a more useful index of reviews and other notices of literature available. In particular, we try here to furnish a classified list of tables which will help readers who wish to use *Mathematical Tables and Other Aids to Computation* as a supplement to existing indices of tables, or who are looking for some particular table.

It is not feasible to repeat enough of the review to make the index compete with such works as "An Index to Mathematical Tables" by Fletcher, Miller, and Rosenhead; however, in order to make the index as convenient as is feasible all entries are given by full title as published in the review indexed. In the review the reader is likely to find a description of the table including:

(1) the precision (a number followed by D indicates the number of digits to the right of the decimal point listed, a number followed by S indicates the number of significant decimal digits listed);

(2) the range of the arguments and their increments (in a standard notation the first number listed is the smallest value of the argument, the standard increment following this smallest value is listed in parentheses, then the largest value attained by steps of this increment is listed, then the next increment is listed in parentheses, and so on);

(3) other pertinent information including uses, related tables and works, and so on.

The index of reviews will not be limited to tables. It is divided into several principal sections in the following order (in this volume):

General Works on Numerical Analysis
Number Theory
Algebra
Analysis
Statistics
Physical Sciences
Automatic Computation

The entries under Analysis are the most voluminous. These include all the mathematical tables which would normally be listed in "An Index of Mathematical Tables" by A. Fletcher, J. C. P. Miller, and L. Rosenhead, Scientific Computing Service, London, 1946, except those in number theory or algebra. In addition, some tables whose entries are restricted by physically measured quantities but which otherwise are analytical in nature may be included.

The classification of the tables indexed under Analysis has followed that of the work cited above through the first digit to the right of the decimal point. No further subdivision has been attempted. In a few cases new subdivisions have been added where no appropriate subdivision was found to exist in the earlier work. The sections are in the same order as the ones listed in the cited work, but the numerical headings have been omitted. The verbal headings of the sections in this work have been retained more or less intact; however, they have been expanded in some cases so that they seem to be self-sufficient without the necessity of including the whole indexing scheme. The Editorial Committee has corresponded with Messrs. Fletcher, Miller, and Rosenhead about this index and we are pleased to acknowledge the cooperation of these authors in our indexing work.

The only major deviations from the usage in the *Index* of Fletcher, Miller, and Rosenhead have resulted from the recent popularity of numbers of nondecimal radix, particularly in connection with automatic computers, and the

addition to the index of approximations, such as the "Approximations for Digital Computers" by C. Hastings, Jr., J. T. Hayward, and J. P. Wong, Jr., Princeton, 1955. References to numbers expressed in common integral bases (such as base two) or expressed symbolically without reference to base (such as some of the approximations cited above) have been made under headings which might have been restricted to decimal headings in the *Index* copied. However, logarithms to bases which are non-decimal have been assigned a special section, not used by Fletcher, Miller, and Rosenhead, and immediately following the section dealing with the corresponding decimal logarithm.

The classification of the tables relating to Number Theory has been carried out in the same way with reference to the "Guide to Tables in the Theory of Numbers" by D. H. Lehmer, The National Research Council, Washington, D. C., 1941. The full detailed classification employed by Lehmer has been used here. Again, only the verbal headings have been repeated, with some augmentation when this seemed desirable to make them stand alone with meaning, and the numbering symbols have been omitted. The order of the listings is the same as that of Lehmer. The Editorial Committee acknowledges the cooperation of Professor Lehmer in granting permission to use his classification.

The classification of tables in Statistics has been done largely by Dr. Bernard Sherman using a complete outline kindly furnished to the Editorial Committee by Professor H. O. Hartley, who is using it in his work on indexing Statistical Tables; we are pleased to acknowledge his help in making his index headings available to us. Tables which might be classified in the Algebra or Analysis sections (some tables which might relate to block designs of experiments and some tables of functions which—like the hypergeometric function—are analytic in origin but have considerable applications in statistics) may not always be included in the Statistics index; however, no attempt has been made to avoid multiple listings, and such omissions are oversights. Tables of quality control have been listed under the miscellaneous heading of the normal distribution.

Adequate listings for the other classifications seem not to be obtainable. At the moment, the number of entries is sufficiently sparse to lead to a belief that the reader will have little difficulty scanning the headings and finding those which seem likely to hold material of interest to him in his particular search.

It should be noted that the headings used are not the only ones which will be used in later indices. In particular, it seems likely that a new category of Social Sciences will be introduced in the next volume.

With regard to many of the more elementary functions, it should be noted that the policy of *MTAC* does not include review of tables of the most common functions. Hence, those included in this index are only those found in more comprehensive books of tables reviewed, and the sections of the index relating to the most common functions should not be considered as even approximately indicating the range of publications including values of these functions during the period reviewed.

The Editorial Committee hopes to be able to assemble a complete index of reviews for the first ten volumes of *MTAC* shortly after the tenth volume has been completed. It will be appreciated if authors of works reviewed and others will suggest proper categories for these works without further solicitation. Com-

ments on the utility of this index and corrections to the present listing will be appreciated if they are sent to the Chairman of the Editorial Committee.

Except for the help cited and considerable advice from Drs. T. H. Southard and M. Melkanoff, the index has been arranged completely by the Chairman of the Editorial Committee, for the first attempt seemed necessarily a job to be undertaken in one place. Mistakes and errors of judgment are wholly his.

C. B. T.

## CORRIGENDA

ROBERT E. GREENWOOD, "Coupon Collector's Test for Random Digits," *MTAC*, v. 9, p. 1, *the displayed relation for* $p_n$ *should read*

$$p_n = \frac{1}{10^{n-1}} \sum_{j=0}^{9} (-1)^j \binom{9}{j} (9-j)^{n-1},$$

*instead of*

$$p_n = \frac{1}{10^{n-1}} \sum_{j=0}^{q} (-1)^j \binom{q}{j} (q-j)^{n-1}.$$

# VOLUME IX

## AUTHOR INDEX

**Papers and Technical Notes**

**[Short papers and notes are marked (N) in this index]**


| *Author* | *Title* | *Page* |
|---|---|---|
| ARCHIBALD, R. C. | Wolfram, Vega, and Thiele (N) | 21 |
| | The First Published Table of Logarithms to the Base Ten (N) | 62–63 |
| ATKINSON, CYRIL | Polynomial Root Solving on the Electronic Differential Analyser | 139–143 |
| BARBER, S. W. | See M. DANK | |
| BERGMAN, STEFAN | Tables for the Determination of Fundamental Solutions of Equations in the Theory of Compressible Fluids | 8–14 |
| BOOTH, A. D. | A Note on Approximating Polynomials for Trigonometric Functions (N) | 21–23 |
| BRENNER, J. L. & G. W. REITWIESNER | Remark on Determination of Characteristic Roots by Iteration (N) | 117–118 |
| CLENSHAW, C. W. | A Note on the Summation of Chebyshev Series (N) | 118–120 |
| COMÉT, STIG | Notations for Partitions | 143–146 |
| DANK, M. & S. W. BARBER | The Specific Heat Function for a Two-Dimensional Continuum | 191–194 |
| FETTIS, HENRY E. | Numerical Calculation of Certain Definite Integrals by Poisson's Summation Formula | 85–92 |
| GOOD, I. G. | Conjectures Concerning the Mersenne Numbers (N) | 120–121 |
| GREENSTADT, J. | A Method for Finding Roots of Arbitrary Matrices | 47–52 |
| GREENWOOD, ROBERT E. | Coupon Collector's Test for Random Digits | 1–5 |
| | Corrigendum | 229 |
| | Table Erratum 246 | 224 |
| HALL, MARSHALL, JR. & J. D. SWIFT | Determination of Steiner Triple Systems of Order 15 | 146–152 |
| HAMMER, PRESTON C. & JACK W. HOLLINGSWORTH | Trapezoidal Methods of Approximating Solutions of Differential Equations | 92–96 |
| HAMMER, PRESTON C. | Iterative Procedures for Taking Roots Based on Square Roots (N) | 68 |
| HERBST, EUGENE H., N. METROPOLIS, & MARK B. WELLS | Analysis of Problem Codes on the MANIAC | 14–20 |
| HO, KUO-CHU | Double Interpolation Formulae and Partial Derivatives in Terms of Finite Differences | 52–62 |
| HOLLINGSWORTH, JACK W. | See PRESTON C. HAMMER | |
| JACCHIA, LUIGI | On the Numerical Integration of Functions Tabulated in Logarithmic Form (N) | 63–65 |
| JEENEL, J. | See S. C. NICHOLSON | |
| LA FARA, R. L. | Modification of a Method for Calculating Inverse Trigonometric Functions (N) | 25 |
| LIPTON, S. | A Note on the Electronic Computer at Rothamsted (N) | 69–70 |
| LOTKIN, MARK | A Set of Test Matrices | 153–161 |
| METROPOLIS, N. | See EUGENE H. HERBST | |

| *Author* | *Title* | *Page* |
|---|---|---|
| NEUMANN, JOHN VON & BRYANT TUCKERMAN | Continued Fraction Expansion of $2^{\frac{1}{3}}$ (N) | 23–24 |
| NICHOLSON, S. C. & J. JEENEL | Some Comments on a NORC Computation of $\pi$ | 162–164 |
| OSTROWSKI, A. M. | Note on a Logarithm Algorithm (N) | 65–68 |
| POSTLEY, J. A. | A Method for the Evaluation of a System of Boolean Algebraic Equations | 5–8 |
| REITWIESNER, G. W. | See J. L. BRENNER | |
| RILEY, JAMES D. | Solving Systems of Linear Equations with a Positive Definite, Symmetric, but Possibly Ill-conditioned Matrix | 96–101 |
| SALZER, HERBERT E. | Orthogonal Polynomials Arising in the Numerical Evaluation of Inverse Laplace Transforms | 164–177 |
| SHELDON, JOHN W. | On the Numerical Solution of Elliptic Difference Equations | 101–102 |
| SNYDER, JAMES W. | On the Improvement of the Solutions to a Set of Simultaneous Linear Equations using the ILLIAC | 177–184 |
| SWIFT, J. D. | See MARSHALL HALL, JR. | |
| TUCKERMAN, BRYANT | See JOHN VON NEUMANN | |
| WELLS, MARK B. | See EUGENE H. HERBST | |
| WILKINSON, J. H. | The Use of Iterative Methods for Finding the Latent Roots and Vectors of Matrices | 184–191 |
| WORSLEY, BEATRICE H. | Solutions of a Nonlinear Differential Equation Arising in the Theory of Diffusion Flames | 112–116 |
| ZONDEK, B. | The Values of $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{1}{3})$ and their Logarithms Accurate to 28 Decimals (N) | 24–25 |


## INDEX OF REVIEWS BY AUTHOR OF WORK REVIEWED


| *Author* | *Review Number* | *Page* |
|---|---|---|
| Abdel-Aty, S. H. | 9 | 30 |
| Abdel-Aty, S. H. | 109 | 218 |
| Abramowitz, Milton | 25 | 38 |
| Admiralty Res. Lab. | 32 | 71 |
| Admiralty Res. Lab. | 41 | 76 |
| Aeronautical Res. Coun. | 52 | 80 |
| Akademiia Nauk SSSR | 39 | 74 |
| Akademiia Nauk SSSR | 40 | 75 |
| Bechhofer, R. E. | 16 | 34 |
| Bechhofer, R. E. | 17 | 34 |
| Berghuis, J. | 28 | 39 |
| Berghuis, J. | 120 | 223 |
| Berry, James G. | 27 | 39 |
| Broadbent, S. R. | 113 | 220 |
| Butler, Terrence | 50 | 79 |
| Cadwell, J. H. | 10 | 31 |
| Cadwell, J. H. | 112 | 219 |
| Chowdhury, S. B. | 114 | 220 |
| Clunie, J. | 26 | 38 |
| Cohen, A. C., Jr. | 7 | 29 |
| Computation Dept., Mathematisch Centrum | 29 | 40 |
| Cornfield, Jerome | 111 | 219 |
| Cox, D. R. | 115 | 221 |
| Daniels, H. E. | 98 | 213 |
| David, F. W. | 96 | 211 |
| David, H. A. | 15 | 33 |
| David, H. A. | 105 | 216 |
| David, H. A. | 116 | 222 |
| Dijkstra, E. W. | 89 | 202 |
| Dixon, W. J. | 102 | 214 |
| Duijvestijn, A. J. W. | 120 | 223 |
| Duncan, A. J. | 107 | 217 |
| Dunnett, C. W. | 18 | 34 |
| Eason, G. | 68 | 132 |
| Epstein, Benjamin | 99 | 213 |
| Flügge, W. | 57 | 123 |
| Foster, F. G. | 97 | 212 |
| Fox, L. | 37 | 73 |
| Frankel, S. | 35 | 72 |
| Freudenthal, A. M. | 110 | 218 |
| Gawlik, H. J. | 117 | 222 |

| Author | Review Number | Page |
|---|---|---|
| Gellman, H. | 42 | 76 |
| Gloden, A. | 1 | 26 |
| Gloden, Albert | 60 | 125 |
| Gloden, Albert | 61 | 126 |
| Greenhouse, Samuel W. | 111 | 219 |
| Grubbs, Frank E. | 36 | 72 |
| Grubbs, F. E. | 88 | 200 |
| Gumbel, E. J. | 110 | 218 |
| Halperin, Max | 111 | 219 |
| Hartley, H. O. | 15 | 33 |
| Hartley, H. O. | 95 | 205 |
| Hartley, H. O. | 105 | 216 |
| Hastings, Cecil, Jr. | 56 | 121 |
| Hayward, Jeanne T. | 56 | 121 |
| Hiester, Nevin K. | 51 | 79 |
| Hill, G. W. | 72 | 134 |
| Hirago, Yoshihiko | 101 | 214 |
| Hruska, Václav | 71 | 133 |
| IBM | 76 | 136 |
| Iyer, P. V. K. | 8 | 30 |
| Jacobson, Arvid W. | 54 | 81 |
| Johnson, N. L. | 96 | 211 |
| König, R. J. | 73 | 135 |
| Lang, E. D. | 24 | 37 |
| Lehmer, Emma | 80 | 196 |
| Lewis, T. | 108 | 217 |
| Lindley, D. V. | 4 | 26 |
| Los Alamos Scientific Lab. | 74 | 135 |
| Lösch, Friedrich | 78 | 195 |
| McKenzie, K. I. | 69 | 133 |
| Mathematesche Centrum | 29 | 40 |
| Miksa, Francis L. | 81 | 197 |
| Miksa, Francis L. | 82 | 197 |
| Miksa, Francis L. | 83 | 198 |
| Miksa, Francis L. | 85 | 198 |
| Miller, J. C. P. | 4 | 26 |
| Miller, J. C. P. | 31 | 70 |
| Mock, Owen R. | 79 | 196 |
| Mock, Owen R. | 90 | 203 |
| Mock, Owen R. | 118 | 223 |
| Morimura, Hidenori | 101 | 214 |
| Murray, F. J. | 88 | 200 |
| NBS | 23 | 37 |
| NBS | 33 | 71 |
| NBS | 38 | 74 |
| NBS | 58 | 124 |
| NBS | 64 | 127 |
| NBS | 65 | 129 |
| NBS | 75 | 135 |
| Nelms, Ann T. | 30 | 40 |
| Nelson, E. | 35 | 72 |
| Noble, B. | 68 | 132 |
| Ondrejka, Rudolph | 2 | 26 |
| Opler, Ascher | 51 | 79 |
| Page, E. S. | 103 | 215 |
| Pearcey, T. | 72 | 134 |
| Pearson, E. S. | 95 | 205 |
| Pearson, E. S. | 105 | 216 |
| Pohlhausen, Karl | 50 | 79 |
| Pollak, L. W. | 34 | 72 |
| Porter, R. J. | 3 | 26 |
| Porter, R. J. | 62 | 126 |
| Porter, R. J. | 84 | 198 |
| Purday, H. F. P. | 66 | 131 |
| Programming Research Group, IBM | 76 | 136 |
| Rabinowitz, Philip | 25 | 38 |
| Rao, A. S. P. | 8 | 30 |
| Robacker, John T. | 119 | 223 |
| Romig, H. G. | 5 | 28 |
| Rosenbaum, S. | 12 | 32 |
| Rothman, M. | 43 | 77 |
| Rothman, M. | 44 | 77 |
| Rothman, M. | 45 | 77 |
| Rothman, M. | 46 | 78 |
| Rothman, M. | 47 | 78 |
| Rothman, M. | 48 | 78 |
| Rothman, M. | 49 | 78 |
| Rothman, M. | 69 | 133 |
| RSMTC | 31 | 70 |
| Ruben, H. | 13 | 32 |
| Rubenstein, Albert H. | 77 | 136 |
| Rushton, S. | 24 | 37 |
| Ryan, R. D. | 72 | 134 |
| Salzer, H. E. | 91 | 203 |
| Salzer, H. E. | 92 | 204 |
| Salzer, Herbert E. | 93 | 204 |
| Salzer, Herbert E. | 94 | 205 |
| Sengupta, J. M. | 19 | 35 |
| Simon, Leslie E. | 36 | 72 |
| Sneddon, I. N. | 68 | 132 |
| Snow, Chester | 70 | 133 |
| Sobel, Milton | 17 | 34 |
| Sobel, Milton | 18 | 34 |
| Spitz, J. C. | 20 | 35 |
| Sterne, T. E. | 104 | 216 |
| Stevens, W. L. | 6 | 28 |

| Author | Review Number | Page |
|---|---|---|
| Stiefel, Eduard L. | 87 | 199 |
| Stoker, J. J. | 88 | 200 |
| Stuart, A. | 97 | 212 |
| Tadman, K. G. | 53 | 81 |
| Tamura, Takayuki | 63 | 126 |
| Terpstra, T. J. | 14 | 33 |
| Thompson, Wm. R. | 22 | 36 |
| Tocher, K. D. | 67 | 132 |
| Tsao, C. K. | 100 | 213 |
| Tucker, Jean | 42 | 76 |
| Uhler, Horace S. | 55 | 121 |
| Vinogradov, I. M. | 59 | 125 |
| Watanabe, Hisao | 101 | 214 |
| Weibull, Martin | 106 | 217 |
| Weiler, H. | 11 | 31 |
| Wheelon, Albert D. | 119 | 223 |
| Whitfield, J. W. | 21 | 36 |
| Wijngaarden, A. Van | 89 | 202 |
| Wong, James P., Jr. | 56 | 121 |
| Woodward, John | 7 | 29 |
| Zavrotsky, Andres | 86 | 198 |
| Zalokar, Julia | 111 | 219 |


## GENERAL WORKS ON NUMERICAL ANALYSIS

### Generalities of Numerical Analysis

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |

### Approximation

| | | | |
|---|---|---|---|
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| EDUARD L. STIEFEL | *Kernel Polynomials in Linear Algebra and their Numerical Applications* | 87 | 199–200 |

### Finite Differences

| | | | |
|---|---|---|---|
| H. F. P. PURDAY | *Linear Equations in Applied Mechanics* | 66 | 131 |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |

### Numerical Integration Summation

| | | | |
|---|---|---|---|
| K. D. TOCHER | "The application of automatic computers to sampling experiments" | 67 | 132 |
| H. E. SALZER | "A simple method for summing certain slowly convergent series" | 91 | 203–204 |

### Linear Equations

| | | | |
|---|---|---|---|
| NBS Applied Mathematics Series No. 29 | *Simultaneous Linear Equations and the Determination of Eigenvalues* | 64 | 127 |
| NBS Applied Mathematics Series No. 39 | *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues* | 65 | 129 |
| H. F. P. PURDAY | *Linear Equations in Applied Mathematics* | 66 | 131 |
| EDUARD L. STIEFEL | *Kernel Polynomials in Linear Algebra and their Numerical Applications* | 87 | 199–200 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Eigenvalue and Eigenvector problems** | | |
| NBS Applied Mathematics Series No. 29 | *Simultaneous Linear Equations and the Determination of Eigenvalues* | 64 | 127 |
| NBS Applied Mathematics Series No. 39 | *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues* | 65 | 129 |
| EDUARD L. STIEFEL | *Kernel Polynomials in Linear Algebra and their Numerical Applications* | 87 | 199–200 |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Linear Programming and Games** | | |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Higher Degree Algebraic Equations** | | |
| ANDRES ZAVROTSKY | *Tablas para la Formacion de las Ecuaciones Cubicas* | 86 | 198–199 |
| | **Ordinary Differential Equations** | | |
| H. F. P. PURDAY | *Linear Equations in Applied Mechanics* | 66 | 131 |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Partial Differential Equations** | | |
| H. F. P. PURDAY | *Linear Equations in Applied Mechanics* | 66 | 131 |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Integral Equations** | | |
| H. F. P. PURDAY | *Linear Equations in Applied Mechanics* | 66 | 131 |
| | **Special Functions** | | |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Sampling, Generating Random Variates, Monte Carlo** | | |
| K. D. TOCHER | "The application of automatic computers to sampling experiments" | 67 | 132 |
| | **General Tables** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |


## TABLES FROM NUMBER THEORY


| | | | |
|---|---|---|---|
| | **Diophantine Equations of form $x^2 \pm Dy^2 = \pm N$** | | |
| FRANCIS L. MIKSA | *Table of Quadratic Partitions $x^2 + y^2 = N$* | 83 | 198 |
| | **Solutions of Diophantine Equations of Second Degree in more than Two Unknowns, Rational Triangles** | | |
| FRANCIS L. MIKSA | *"A table of integral solutions of $a^2 + b^2 + c^2 = r^2$"* | 81 | 197 |
| FRANCIS L. MIKSA | *A Table of Integral Solutions of $A^2 + B^2 + C^2 = R^2$ for all odd $R$ from $R = 3$ to $R = 325$, in two sets, $R = 1$ to 207 and an extension from $R = 207$ to $R = 325$.* | 82 | 197 |
| | **Perfect and Amicable Numbers and their Generalizations** | | |
| RUDOLPH ONDREJKA | List of the First 17 Perfect Numbers | 2 | 26 |
| | **Tables of Binomial Coefficients** | | |
| RSMTC | *Table of Binomial Coefficients* | 31 | 70 |
| | **Tables of Primitive Roots** | | |
| I. M. VINOGRADOV | *Elements of Number Theory* | 59 | 125 |
| | **Tables of Exponents and Residue-indices** | | |
| I. M. VINOGRADOV | *Elements of Number Theory* | 59 | 125 |
| | **Solutions of Special Binomial Congruences** | | |
| ALBERT GLODEN | *Table des Solutions de la Congruence $x^4 + 1 \equiv 0 \pmod{p^2}$ pour $4000 < p < 6000$* | 60 | 125 |
| ALBERT GLODEN | *Table des Solutions Minima de la Congruence $x^4 + 1 \equiv \pmod{p^2}$ pour $6000 < p < 10000$* | 61 | 126 |
| | **Tables of Factors of Numbers of Special Form** | | |
| A. GLODEN | *Table des factorisation des Nombres $N^4 + 1$ dans l'intervalle $10000 < N \leq 20000$* | 1 | 26 |
| HORACE S. UHLER | *"Hamartiexéresis* as applied to tables involving logarithms" | 55 | 121 |
| | **Lists of Consecutive Primes** | | |
| I. M. VINOGRADOV | *Elements of Number Theory* | 59 | 125 |
| | **Tables of solutions of non-binomial congruences of degree $\geq 3$** | | |
| EMMA LEHMER | "On the cyclotomic numbers of order sixteen" | 80 | 196–197 |
| | **Non-linear Forms, their Classes and Class Numbers** | | |
| R. J. PORTER | *A List of Groups and Series to serve for Computations of Irregular Negative Determinants of Exponent 3n* | 3 | 26 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| R. J. PORTER | *List of Irregular Determinants of Exponent 3n* [$80000 < -D < 150000$] | 62 | 126 |
| R. J. PORTER | *Irregular Negative Determinants of Exponent 3n with their critical classes* | 84 | 198 |


## TABLES FROM ALGEBRA

### Tables involving Marks from Algebras or Arithmetics other than the usual Number Systems


| | | | |
|---|---|---|---|
| TAKAYUKI TAMURA | "Note on finite semigroups and determination of semigroups of order 4" | 63 | 126 |


## TABLES FROM ANALYSIS

### Squares


| | | | |
|---|---|---|---|
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Cubes


| | | | |
|---|---|---|---|
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Higher Integral Powers


| | | | |
|---|---|---|---|
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Square Roots


| | | | |
|---|---|---|---|
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Fractional Powers of Real Numbers


| | | | |
|---|---|---|---|
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |


### Powers of Complex Numbers


| | | | |
|---|---|---|---|
| ADMIRALTY RESEARCH LABORATORY | "Tables of $\eta = \sqrt{(m^2 + \sqrt{-1})}$" | 32 | 71 |


### Factorials, etc.


| | | | |
|---|---|---|---|
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Binomial Coefficients


| | | | |
|---|---|---|---|
| RSMTC | *Table of Binomial Coefficients* | 31 | 70 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |


### Generalized Bernoulli and Euler Numbers and Polynomials. Differences and Derivatives of zero, etc. Coefficients in Formulas connecting Derivatives and Integrals with Differences


| | | | |
|---|---|---|---|
| FRANCIS L. MIKSA | *Stirling Numbers of the Second Kind* | 85 | 198 |

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| | **Constants involving π** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Constants involving *e*. Circular and Hyperbolic functions of Constant Arguments** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Algebraic Constants. Roots of Algebraic Equations** | | |
| ANDRES ZAVROTSKY | *Tablas para la Formacion de las Ecuaciones Cubicas* | 86 | 198–199 |
| | **Conversion Tables** | | |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Fundamental Tables of Logarithms to Non-decimal Integral Bases** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Natural Trigonometrical Functions with Argument in Radian** | | |
| NBS Applied Mathematics Series No. 36 | *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments* | 33 | 71 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| NBS Applied Mathematics Series No. 43 | *Tables of Sines and Cosines for Radian Arguments* | 58 | 124 |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–195 |
| | **Natural Trigonometrical Functions with Argument in Degrees and Decimals** | | |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Natural Trigonometrical Functions with Centesimal Argument** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Inverse Circular Functions** | | |
| W. L. STEVENS | "Tables of the angular transformation" | 6 | 28 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| | **Natural Values of $e^x$** | | |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |
| E. S. PEARSON<br>H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Quantities (Natural and Logarithmic) connected with $e^{\pm x}$** | | |
| CECIL HASTINGS, JR.,<br>JEANNE T. HAYWARD,<br>JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Natural Values of Hyperbolic Functions** | | |
| NBS<br>Appl. Math.<br>Ser. No. 36 | *Tables of Circular and Hyperbolic Sines for Radian Arguments* | 33 | 71 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| FRIEDRICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |
| | **Natural Logarithms of Numbers** | | |
| CECIL HASTINGS, JR.,<br>JEANNE T. HAYWARD,<br>JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| FRIEDERICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |
| E. S. PEARSON<br>H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Natural Logarithms of Reciprocals, Square Roots and other Functions** | | |
| CECIL HASTINGS, JR.,<br>JEANNE T. HAYWARD,<br>JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Inverse Hyperbolic Functions** | | |
| FRIEDERICH LÖSCH | *Siebenstellige Tafeln der elementaren transzendenten Funktionen* | 78 | 195–196 |
| E. S. PEARSON<br>H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Miscellaneous Functions of Circular and Hyperbolic Functions and Inverse Functions** | | |
| AECD-3497<br>S. FRANKEL,<br>E. NELSON | "Methods of treatment of displacement integral equations" | 35 | 72 |
| | **Exponential Integrals Ei(x)** | | |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| | **Exponential Integrals. $-\mathrm{Ei}(-x)$** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Sine Integral** | | |
| AKADEMIIÂ NAUK SSSR | *Tablitsy Integralnogo sinusa i kosinusa [Tables of the sine and cosine integral]* | 39 | 74 |
| NBS Applied Mathematics Series No. 32 | *Table of Sine and Cosine Integrals for Arguments from 10 to 100* | 23 | 37 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Cosine Integral** | | |
| NBS Applied Mathematics Series No. 32 | *Table of Sine and Cosine Integrals for Arguments from 10 to 100* | 23 | 37 |
| AKADEMIIÂ NAUK SSSR | *Tablitsy Integralnogo sinusa i kosinusa [Tables of the sine and cosine integral]* | 39 | 74 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **The Gamma Function** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **$\mathrm{Log}_{10}\,\Gamma(1+x)$** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Gamma Function for Complex Arguments** | | |
| NBS Applied Math. Series No. 34 | *Table of the Gamma Function for Complex Arguments* | 38 | 74 |
| | **Incomplete Beta Function** | | |
| E. S. PEARSON H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **The Error Integral. Tables of the Ordinate (the Integrand)** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Tables of the Error Integral** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **The Error Integral. Ratio of Integral to Ordinate** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| H. J. GAWLIK | *A Table of a Function related to the Error Function* | 117 | 222 |
| | **The Error Integral. Inverse Tables** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Moments. Chi-squared Tables** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **The Error Integral with Complex Argument** | | |
| T. LAIBLE | "Hohenkarte des Fehlerintegrals" | E 244 | 136 |
| | **Legendre Polynomials and their Applications** | | |
| K. G. TADMAN | *Tables of Flow Functions for Bodies of Revolution in Circular Tunnels and Jets* | 53 | 81 |
| | **Derivatives of Legendre Polynomials and their Applications** | | |
| K. G. TADMAN | *Tables of Flow Functions for Bodies of Revolution in Circular Tunnels and Jets* | 53 | 81 |
| | **Legendre and Associated Functions of Fractional or Complex Order** | | |
| CHESTER SNOW | "Formulas for computing capacitance and inductance" | 70 | 133 |
| | **Bessel Functions of the First Kind: Integral Order** | | |
| L. FOX | *A Short Table for Bessel Functions of Integer Order and Large Arguments* | 37 | 73 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| OWEN R. MOCK | *Table of Bessel Functions at Multiples of $\pi/2$* | 118 | 223 |
| | **Bessel Functions of the Second Kind: Integral Order** | | |
| L. FOX | *A Short Table for Bessel Functions of Integer Order and Large Arguments* | 37 | 73 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Modified Bessel Functions of the First Kind: Integral Order** | | |
| L. FOX | *A Short Table for Bessel Functions of Integer Order and Large Arguments* | 37 | 73 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| | **Modified Bessel Functions of the Second Kind: Integral Order** | | |
| L. Fox | *A Short Table for Bessel Functions of Integer Order and Large Arguments* | 37 | 73 |
| W. Flügge | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Kelvin Functions of the First Kind** | | |
| W. Flügge | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Kelvin Functions of the Second Kind** | | |
| W. Flügge | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Kelvin Functions: Modulus and Phase** | | |
| E. Jahnke, F. Emde | *Tables of Functions* | E 243 | 42 |
| | **The Airy and Allied Integrals** | | |
| M. Rothman | *Table of* $\mathrm{Bi}(+x)$ *for* $x = 0.00(0.01)2.00$ and $\mathrm{Bi}(-x)$ *for* $x = 0.00(0.01)10.00$ | 45 | 77 |
| K. I. McKenzie, M. Rothman | *Table of* $\mathrm{Bi}'(+x)$ *for* $x = 0(0.01)2$ *and* $\mathrm{Bi}'(-x)$ *for* $x = 0(0.01)10$ | 69 | 133 |
| | **Fresnel Integrals** | | |
| Akademiia Nauk SSSR | *Tablitsy Integralov Frenelya* [*Tables of Fresnel Integrals*] | 40 | 75 |
| W. Flügge | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Integrals involving Bessel Functions** | | |
| M. Rothman | *Table of the integrals of* $\mathrm{Ai}(\pm x)$ *for* $x = 0.00(0.01)2.00$ *and* $x = 0.0(0.1)10.0$ | 44 | 77 |
| M. Rothman | *Table of the integrals of* $\mathrm{Bi}(\pm x)$ for $x = 0.00(0.01)2.00$ *and of* $\mathrm{Bi}(-x)$ *for* $x = 0.0(0.1)10.0$ | 46 | 78 |
| Terence Butler, Karl Pohlhausen | *Tables of Definite Integrals involving Bessel Functions of the First Kind* | 50 | 79 |
| Ascher Opler, Nevin K. Hiester | *Tables for Predicting the Performance of Fixed Bed Ion Exchange and Similar Mass Transfer Processes* | 51 | 79 |
| Cecil Hastings, Jr., Jeanne T. Hayward, James P. Wong, Jr. | *Approximations for Digital Computers* | 56 | 121 |
| G. Eason, B. Noble, I. N. Sneddon | "On certain integrals of Lipschitz-Hankel type involving products of Bessel functions" | 68 | 132 |
| Albert D. Wheelon, John T. Robacker | *A Table of Integrals Involving Bessel Functions* | 119 | 223 |
| | **Complete Elliptic Integrals of the First and Second Kinds** | | |
| Cecil Hastings, Jr., Jeanne T. Hayward, James P. Wong, Jr. | *Approximations for Digital Computers* | 56 | 121 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| | **Incomplete Elliptic Integrals of the First and Second Kinds** | | |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| | **Hypergeometric Functions** | | |
| S. RUSHTON | "On the confluent hypergeometric function $M(\alpha, \gamma, x)$" | 24 | 37 |
| MILTON ABRAMOWITZ, PHILIP RABINOWITZ | "Evaluation of Coulomb wave functions along the transition line" | 25 | 38 |
| | **Debye Functions, Radiation Integrals, Fermi-Dirac Functions, etc.** | | |
| J. CLUNIE | "On Bose-Einstein functions" | 26 | 38 |
| ANN T. NELMS | "Graphs of the Compton energy-angle relationship and the Klein-Nishina formula from 10 Kev to 500 Mev" | 30 | 40 |
| A. J. W. DUIJVESTIJN, J. BERGHUIS | *The Computation and the Expansion of Some Triple Integrals Originating from the Theory of Cosmic Rays* | 120 | 223–224 |
| | **Miscellaneous Integrals Involving Circular and Exponential Functions** | | |
| L. BERGHUIS | *A Table of Some Integrals* | 28 | 39 |
| ADMIRALTY RESEARCH LABORATORY | "Table of $\lvert I\rvert = \lvert\int_0^{\varphi} \sec\theta e^{(a+ib)\theta} d\theta\rvert$" | 41 | 76 |
| H. GELLMAN, JEAN TUCKER | "Tables of the functions $D_0(x)$ and $D_1(x)$" | 42 | 76 |
| M. ROTHMAN | "Tables of the integrals and differential coefficients of $\mathrm{Gi}(+x)$ and $\mathrm{Hi}(-x)$" | 43 | 77 |
| M. ROTHMAN | *Table of* $\mathrm{Gi}(x)$ *and its derivative for* $x = 0.0(0.1)25(1)75$ | 47 | 78 |
| M. ROTHMAN | *Table of* $\mathrm{Hi}(-x)$ *and its derivative for* $x = 0.0(0.1)25(1)75$ | 48 | 78 |
| M. ROTHMAN | *Tables of the integrals of* $\mathrm{Gi}(x)$ *and* $\mathrm{Hi}(-x)$ *for* $x = 0.0(0.1) - 20.0$ | 49 | 78 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Miscellaneous Generalizations of Common Functions** | | |
| Staff of the Computation Department of Mathematisch Centrum | *Table of Polylogarithms, Report R24, Part I: Numerical Values* | 29 | 40 |
| | **Interpolation Coefficients and Formulas** | | |
| E. W. DIJKSTRA, A. VAN WIJNGAARDEN | *Table of Everett's Interpolation Coefficients* | 89 | 202–203 |
| HERBERT E. SALZER | *Formulas for Calculating Fourier Coefficients* | 93 | 204–205 |
| HERBERT E. SALZER | *Formulas for Inverse Osculatory Interpolation* | 94 | 205 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| | **Coefficients for Integration and Summation using Ordinates** | | |
| H. E. SALZER | "A simple method for summing certain slowly convergent series" | 91 | 203–204 |
| H. E. SALZER | "Equally weighted quadrature formulas over semi-infinite and infinite intervals" | 92 | 204 |
| | **Coefficients for Integration and Summation, using Differences** | | |
| OWEN R. MOCK | *Fifty Everett Integration Coefficients* | 90 | 203 |
| | **Curve-Fitting, etc.** | | |
| HERBERT E. SALZER | *Formulas for Calculating Fourier Coefficients* | 93 | 204–205 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Harmonic Analysis—Sines and Cosines** | | |
| L. W. POLLAK | *Eight-place Supplement to Harmonic Analysis and Synthesis Schedules for Three to One Hundred Equidistant Values of Empiric Functions* | 34 | 72 |
| OWEN R. MOCK | *A Table of Sines and Cosines of* $\frac{\pi}{2} \cdot \frac{h}{256}$ | 79 | 196 |


## TABLES FROM STATISTICS


| | | | |
|---|---|---|---|
| | **Tables with the Normal Deviate as Arguments** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| W. FLÜGGE | *Four-place Tables of Transcendental Functions* | 57 | 123 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Tables with the Normal Area as Argument, including percentage points and probits** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Rankings of Normal Samples or Order-statistics, Median** | | |
| H. RUBEN | "On the moments of order statistics in samples from normal populations" | 13 | 32 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| | **Measures of Location and Measures of Normal Dispersion other than the Standard Deviation** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| J. H. CADWELL | "The statistical treatment of mean deviation" | 10 | 31 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| H. A. DAVID, H. O. HARTLEY, E. S. PEARSON | "The distribution of the ratio, in a single normal sample, of range to standard deviation" | 105 | 216 |
| A. J. DUNCAN | "The use of ranges in comparing variabilities" | 107 | 217 |
| MAX HALPERIN, SAMUEL W. GREENHOUSE, JEROME CORNFIELD, JULIA ZALOKAR | "Tables of percentage points for the Studentized maximum absolute deviate in normal samples" | 111 | 219 |
| | **Tests of Normality Criteria** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Miscellanea (Normal Distribution)** | | |
| A. C. COHEN, JR., JOHN WOODWARD | "Tables of Pearson-Lee-Fisher functions of singly truncated normal distributions" | 7 | 29 |
| H. WEILER | "A new type of control chart limits for means, ranges, and sequential runs" | 11 | 31 |
| R. E. BECHHOFER | "A single-sample multiple decision procedure for ranking means of normal populations with known variances" | 16 | 34 |
| R. E. BECHHOFER, MILTON SOBEL | "A single-sample multiple decision procedure for ranking variances of normal populations" | 17 | 34 |
| C. W. DUNNETT, MILTON SOBEL | "A bivariate generalization of Student's $t$-distribution with tables for certain special cases" | 18 | 34–35 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| E. S. PAGE | "Control charts for the mean of a normal population" | 103 | 215–216 |
| S. B. CHOWDHURY | "The most powerful unbiased critical regions and the shortest unbiased confidence intervals associated with the distribution of the classical $D^2$-statistic" | 114 | 220–221 |
| H. J. GAWLIK | *A Table of a Function Related to the Error Function* | 117 | 222 |
| | **Area of Incomplete $\Gamma$-function** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Ordinates of $\chi^2$-Distribution and Poisson Frequencies** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| | **Percentage Points of $\chi^2$-Distribution, Percentage Points of Poisson Series** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| T. LEWIS | "99.9 and 0.1% of the $\chi^2$-Distribution" | 108 | 217–218 |
| | **Laguerre Series, Gram-Charlier Series B, Non-Central $\chi^2$ Series, Fractional Degrees of Freedom** | | |
| S. H. ABDEL-ATY | "Approximate formulae for the percentage points of the non-central $\chi^2$-Distribution" | 109 | 218 |
| | **Cumulative Binomial Series, Tail Area of $I_x(a, b)$** | | |
| H. G. ROMIG | *50–100 Binomial Tables* | 5 | 28 |
| LESLIE E. SIMON, FRANK E. GRUBBS | *Tables of the Cumulative Binomial Probabilities* | 36 | 72 |
| | **Individual Frequencies of Binomial Distribution** | | |
| H. G. ROMIG | *50–100 Binomial Tables* | 5 | 28 |
| RSMTC | *Table of Binomial Coefficients* | 31 | 70 |
| LESLIE E. SIMON, FRANK E. GRUBBS | *Tables of the Cumulative Binomial Probabilities* | 36 | 72 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Percentage Points of $I_x(a, b)$, Confidence Belts for Binomial Distribution** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| T. E. STERNE | "Some remarks on confidence or fiducial limits" | 104 | 216 |
| | **Tables of $t$-Distribution** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| J. M. SENGUPTA | "Significance level of $\Sigma x^2/(\Sigma x)^2$ based on Student's distribution" | 19 | 35 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| MARTIN WEIBULL | "The distribution of $t$- and $F$-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means" | 106 | 217 |
| | **Tables of $F$-Distribution and Multivariate Variance Tests** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| MARTIN WEIBULL | "The distribution of *t*- and *F*-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means" | 106 | 217 |
| | **Hypergeometric Series** | | |
| H. G. ROMIG | *50–100 Binomial Tables* | 5 | 28 |
| | **Miscellaneous Discrete Distributions** | | |
| S. H. ABDEL-ATY | "Tables of generalized *k*-statistics" | 9 | 30 |
| J. C. SPITZ | "Matching in psychology" | 20 | 35–36 |
| J. W. WHITFIELD | "The distribution of the difference in total rank value for two particular objects in *m* rankings of *n* objects" | 21 | 36 |
| WM. R. THOMPSON | *Tables of the Four Variable N- and* Psi-*Functions* | 22 | 36 |
| | **Distribution of Ordinary Correlations** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| MARTIN WEIBULL | "The distribution of *t*- and *F*-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means" | 106 | 217 |
| | **Fisher's *r* to *z* and *z* to *r* Transformations** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Rank Correlation and Associated Criteria** | | |
| J. W. WHITFIELD | "The distribution of the difference in total rank value for two particular objects in *m* rankings of *n* objects" | 21 | 36 |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Rankings, Gaps, etc., for Rectangular and other Non-Normal Distributions** | | |
| H. A. DAVID | "The distribution of the range in certain non-normal populations" | 116 | 222 |
| | **Order Statistics from any Continuous Parent** | | |
| H. O. HARTLEY, H. A. DAVID | "Universal bounds for mean range and extreme observation" | 15 | 33 |
| F. W. DAVID, N. L. JOHNSON | "Statistical treatment of censored data. Part I. Fundamental formulae" | 96 | 211–212 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| BENJAMIN EPSTEIN | "Tables for the distribution of the number of the exceedances" | 99 | 213 |
| J. H. CADWELL | "The probability integral of range for samples from a symmetrical unimodal population" | 112 | 219 |
| | **Non-Parametric Two Sample and Analysis of Variance Randomization Tests** | | |
| S. ROSENBAUM | "Tables for a nonparametric test of location" | 12 | 32 |
| C. K. TSAO | "An extension of Massey's distribution of the maximum deviation between two-sample cumulative step functions" | 100 | 213–214 |
| | **Runs** | | |
| P. V. K. IYER, A. S. P. RAO | "Theory of the probability distribution of runs in a sequence of observations" | 8 | 30 |
| | **Other Non-Parametric Tests** | | |
| T. J. TERPSTRA | "The exact probability distribution of the $T$ statistic for testing against trend and its normal approximation" | 14 | 33 |
| F. G. FOSTER, A. STUART | "Distribution-free tests in time-series based on the breaking of records" | 97 | 212 |
| H. E. DANIELS | "A distribution-free test for regression parameters" | 98 | 213 |
| YOSHIHIKO HIRAGO, HIDENORI MORIMURA, HISAO WATANABE | "Tables for three-sample test" | 101 | 214 |
| W. J. DIXON | "Power under normality of several nonparametric tests" | 102 | 214–215 |
| | **Formulas and Tables of Pearson Type Frequency Distributions** | | |
| E. S. PEARSON, H. O. HARTLEY | *Biometrika Tables for Statisticians* | 95 | 205–211 |
| | **Tables of Miscellaneous Continuous Distributions** | | |
| J. CLUNIE | "On Bose-Einstein functions" | 26 | 38–39 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121–123 |
| A. M. FREUDENTHAL, E. J. GUMBEL | "Minimum life in fatigue" | 110 | 218–219 |
| S. R. BROADBENT | "The quotient of a rectangular or triangular and a general variate" | 113 | 220 |
| | **Distribution of Means from Non-Normal Parents** | | |
| D. R. COX | "The mean and coefficient of variation of range in small samples from non-normal populations" | 115 | 221–222 |
| | **Angular Transformations** | | |
| W. L. STEVENS | "Tables of the angular transformation" | 6 | 28 |

| Author/s | Title | Review Number | Page |
|---|---|---|---|
| | **Random Numbers** | | |
| D. V. LINDLEY, J. C. P. MILLER | *Cambridge Elementary Statistical Tables* | 4 | 26–27 |
| | **Random Deviates from Mechanical Devices. Quasi-random Numbers** | | |
| K. D. TOCHER | "The application of automatic computers to sampling experiments" | 67 | 132 |
| | **Random Normal Deviates** | | |
| K. D. TOCHER | "The application of automatic computers to sampling experiments" | 67 | 132 |


## TABLES FROM PHYSICAL SCIENCES

NOTE: Many tables whose calculation was directly instigated by research in physics are purely mathematical in nature and these are classified under Analysis or another appropriate heading.


| | | | |
|---|---|---|---|
| | **Mechanics of Solids** | | |
| JAMES G. BERRY | *Tables of some Functions Related to the Legendre Functions $P_n^{-m}(x)$ and $Q_n(x)$ when n is a complex number* | 27 | 39 |
| | **Fluid Mechanics** | | |
| AERONAUTICAL RESEARCH COUNCIL | *A Selection of Graphs for Use in Calculations of Compressible Airflow* | 52 | 80–81 |
| K. G. TADMAN | *Tables of Flow Functions for Bodies of Revolution in Circular Tunnels and Jets* | 53 | 81 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121 |
| | **Exchanges, Reaction Kinetics, Reaction Rates, etc.** | | |
| ASCHER OPLER, NEVIN K. HIESTER | *Tables for Predicting the Performance of Fixed Bed Ion Exchange and Similar Mass Transfer Processes* | 51 | 79 |
| | **Electricity and Magnetism** | | |
| CHESTER SNOW | "Formulas for computing capacitance and inductance" | 70 | 133 |
| | **Atomic Physics** | | |
| MILTON ABRAMOWITZ, PHILIP RABINOWITZ | "Evaluation of Coulomb wave functions along the transition line" | 25 | 38 |
| | **Nuclear Physics and Scattering (Including Photon Scattering and Compton Effect)** | | |
| ANN T. NELMS | "Graphs of the Compton energy-angle relationship and the Klein-Nishina formula from 10 Kev to 500 Mev" | 30 | 40 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| H. GELLMAN, JEAN TUCKER | "Tables of the functions $D_0(x)$ and $D_1(x)$" | 42 | 76 |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121–123 |
| | **Cosmic Rays** | | |
| A. J. W. DUIJVESTIJN, J. BERGHUIS | "The Computation and the Expansion of some triple integrals originating from the theory of cosmic rays" | 120 | 223–224 |
| | **Statistical Mechanics—Classical and Quantum** | | |
| J. CLUNIE | "On Bose-Einstein functions" | 26 | 38 |
| | **AUTOMATIC COMPUTATION** | | |
| | **Conferences and Symposia on Computing Machines or Calculation** | | |
| ARVID W. JACOBSON, Editor | *Proceedings of the First Conference on Training Personnel for the Computing Machine Field* | 54 | 81 |
| ALBERT H. RUBENSTEIN, Editor | *Coordination, Control, and Financing of Industrial Research* | 77 | 136 |
| F. E. GRUBBS, F. J. MURRAY, J. J. STOKER | *Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* | 88 | 200–202 |
| | **Approximations of Functions for Machine Use** | | |
| CECIL HASTINGS, JR., JEANNE T. HAYWARD, JAMES P. WONG, JR. | *Approximations for Digital Computers* | 56 | 121–123 |
| | **Applications to Statistical Problems** | | |
| K. D. TOCHER | "The application of automatic computers to sampling experiments" | 67 | 132 |
| | **Logic of Coding** | | |
| T. PEARCEY, G. W. HILL, R. D. RYAN | "The effect of interpretive techniques on functional design of computers" | 72 | 134–135 |
| R. J. KÖNIG | *Introduction to FLAC Coding* | 73 | 135 |
| LOS ALAMOS SCIENTIFIC LABORATORY | *Los Alamos Scientific Laboratory, MANIAC* | 74 | 135 |
| NATIONAL BUREAU OF STANDARDS | *Manual of the SWAC Computing System* | 75 | 135–136 |
| | **Interpretive Routines—Automatic Coding** | | |
| T. PEARCEY, G. W. HILL, R. D. RYAN | "The effect of interpretive techniques on functional design of computers" | 72 | 134–135 |

| *Author/s* | *Title* | *Review Number* | *Page* |
|---|---|---|---|
| Los Alamos Scientific Laboratory | *Los Alamos Scientific Laboratory, MANIAC* | 74 | 135 |
| Programming Research Group, Applied Science Division, International Business Machines Corp. | "Specifications for the IBM mathematical FORmula TRANslating system, FORTRAN" | 76 | 136 |
| | **Machine Descriptions—Punched Card** | | |
| Václav Hruska (editor) | *Československá Akad. Věd. Laboratoř matematických strojů. Matematické Stroje (Mathematical Machines)* | 71 | 133–134 |
| | **Machine Descriptions—Electronic** | | |
| Václav Hruska (editor) | *Československá Akad. Věd. Laboratoř matematických strojů. Matematické Stroje (Mathematical Machines)* | 71 | 133–134 |
| R. J. König | *Introduction to FLAC Coding* | 73 | 135 |
| Los Alamos Scientific Laboratory | *Los Alamos Scientific Laboratory MANIAC* | 74 | 135 |
| National Bureau of Standards | *Manual of the SWAC Computing System* | 75 | 135–136 |
| | **Machine Programs and Codes for Specific Problems** | | |
| Václav Hruska (editor) | *Československá Akad. Věd. Laboratoř matematických strojů. Matematické Stroje (Mathematical Machines)* | 71 | 133–134 |
| Los Alamos Scientific Laboratory | *Los Alamos Scientific Laboratory, MANIAC* | 74 | 135 |
| | **Machine Design—Generalities** | | |
| Václav Hruska (editor) | *Československá Akad. Věd. Laboratoř matematických strojů. Matematické Stroje (Mathematical Machines)* | 71 | 133–134 |

## TABLE ERRATA

| *No.* | *Author* | *Title* | *Page* |
|---|---|---|---|
| 242 | R. S. Burington | *Handbook of Mathematical Tables and Formulas* | 41 |
| 243 | E. Jahnke & F. Emde | *Tables of Functions* | 42 |
| 244 | T. Laible | "Hohenkarte des Fehlerintegrals" | 136 |
| 245 | E. Jahnke & F. Emde | *Tables of Functions* | 224 |
| 246 | Robert E. Greenwood | "Coupon Collector's Test for Random Digits" | 224 |
| | | **Also see the following reviews:** | |
| 55 | Horace S. Uhler | "*Hamartiexéresis* as applied to tables involving logarithms" | 121 |
| 59 | I. M. Vinogradov | *Elements of Number Theory* | 125 |
| 80 | E. Lehmer | "On the cyclotomic numbers of order sixteen" | 196–197 |
| 81 | Francis L. Miksa | "A table of integral solutions of $a^2 + b^2 + c^2 = r^2$" | 197 |
| 89 | E. W. Dijkstra & A. Van Wijngaarden | *Table of Everett's Interpolation Coefficients* | 202–203 |

## NOTES

A Conference on Mathematical Tables ..... 42
International Analogy Computation Meeting ..... 44
The Royal Society Depository for Unpublished Mathematical Tables ..... 44
Note on Arrangement of Material ..... 45
Summary of Educational Opportunities in Electronic Computation ..... 82
E. W. Cannon appointed Chief of Applied Mathematics Division of the National Bureau of Standards ..... 83
Policy Committee for Mathematics, NBS Technical Advisory Committee ..... 83
The University of Wisconsin Conference, "The Computing Laboratory in the University" .. 84
International Conference on "Electronic Digital Computers and Information Processing" ... 137
R. C. Archibald ..... 224
Postdoctoral Research Associates of the National Bureau of Standards ..... 225
National Simulation Conference ..... 225
The Second Regional Graduate Summer Session in Statistics held at the University of Florida in June and July 1955 ..... 226
Sponsorship by the Office of Naval Research, the National Bureau of Standards, the National Science Foundation, and various other organizations ..... 226
Index ..... 226

## QUERIES AND REPLIES

Queries and Replies ..... 137

## CORRIGENDA

| *Author* | *Title* | *Page* |
|---|---|---|
| WILKES, M. V. | "A Table of Chapman's grazing incidence integral $Ch(x, \chi)$" .... | 45 |
| BLAKESLEY, T. H. | A Table of Hyperbolic Cosines and Sines ..... | 45 |
| BODEWIG, E. | A Practical Refutation of the Iteration Method for the Algebraic Eigenproblem ..... | 137 |
| WILKES, M. V. | "A Table of Chapman's grazing incidence integral $Ch(x, \chi)$" .... | 137 |
| GREENWOOD, ROBERT E. | "Coupon Collector's Test for Random Digits" ..... | 229 |

## CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences. Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

Q. Astronomy

R. Geodesy

S. Physics, Geophysics, Crystallography

T. Chemistry

U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation

# CONTENTS

## OCTOBER 1955


Polynomial Root Solving on the Electronic Differential Analyser (A Technique for Finding the Real and Complex Roots of a Polynomial using an Electronic Differential Analyser) ..... CYRIL ATKINSON 139

Notations for Partitions ..... STIG COMÉT 143

Determination of Steiner Triple Systems of Order 15 ..... MARSHALL HALL, JR. & J. D. SWIFT 146

A Set of Test Matrices ..... MARK LOTKIN 153

Some Comments on a NORC Computation of $\pi$ ..... S. C. NICHOLSON & J. JEENEL 162

Orthogonal Polynomials Arising in the Numerical Evaluation of Inverse Laplace Transforms ..... HERBERT E. SALZER 164

On the Improvement of the Solutions to a Set of Simultaneous Linear Equations using the ILLIAC ..... JAMES N. SNYDER 177

The Use of Iterative Methods for Finding the Latent Roots and Vectors of Matrices ..... J. H. WILKINSON 184

Technical Notes and Short Papers ..... 191

The Specific Heat Function for a Two-Dimensional Continuum ..... M. DANK & S. W. BARBER 191

A Continued Fraction for $e^x$ ..... N. MACON 194

Reviews and Descriptions of Tables and Books ..... 195

LÖSCH 78, MOCK 79, LEHMER 80, MIKSA 81, 82, 83, PORTER 84, MIKSA 85, ZAVROTSKY 86, STIEFEL 87, GRUBBS, MURRAY, & STOKER 88, DIJKSTRA & VAN WIJNGAARDEN 89, MOCK 90, SALZER 91, 92, 93, 94, PEARSON & HARTLEY 95, DAVID & JOHNSON 96, FOSTER & STUART 97, DANIELS 98, EPSTEIN 99, TSAO 100, HIRAGO, MORIMURA, & WATANABE 101, DIXON 102, PAGE 103, STERNE 104, DAVID, HARTLEY, & PEARSON 105, WEIBULL 106, DUNCAN 107, LEWIS 108, ABDEL-ATY 109, FREUDENTHAL & GUMBEL 110, HALPERIN, GREENHOUSE, CORNFIELD, & ZALOKAR 111, CADWELL 112, BROADBENT 113, CHOWDHURY 114, COX 115, DAVID 116, GAWLIK 117, MOCK 118, WHEELON & ROBACKER 119, DUIJVESTIJN & BERGHUIS 120.

Table Errata ..... 224

JAHNKE & EMDE 245, GREENWOOD 246.

Notes ..... 224

R. C. Archibald ..... 224

Postdoctoral Research Associates of the National Bureau of Standards ..... 225

National Simulation Conference ..... 225

The Second Regional Graduate Summer Session in Statistics held at the University of Florida in June and July 1955 ..... 226

Sponsorship by the Office of Naval Research, the National Bureau of Standards, the National Science Foundation, and various other organizations ..... 226

Index ..... 226

Corrigenda ..... 229

GREENWOOD ..... 229

Indices to volume IX ..... 230

Author Index, Papers and Technical Notes ..... 230

Reviews by Author of Work Reviewed ..... 231

Reviews by Subject ..... 233

General Works on Numerical Analysis ..... 233

Tables from Number Theory ..... 235

Tables from Algebra ..... 236

Tables from Analysis ..... 236

Tables from Statistics ..... 243

Tables from Physical Sciences ..... 248

Automatic Computation ..... 249

Table Errata ..... 250

Notes ..... 251

Queries and Replies ..... 251

Corrigenda ..... 251


LANCASTER PRESS, INC., LANCASTER, PA.